

## 数学 I 期末試験 解答

1.(20 点)

$$V = \iint_D z dx dy = \iint_D (3x^2 + 2xy) dx dy, \quad D: (x-3)^2 + (y-2)^2 \leq 1 \text{ を } x-3=u, y-2=v \text{ とお$$

いて変換する.  $D': u^2 + v^2 \leq 1, |J|=1$  である.

$$V = \iint_{D'} \{3(u+3)^2 + 2(u+3)(v+2)\} dudv \text{ さらに } u = r \cos \theta, v = r \sin \theta \text{ において極座標変換}$$

を使う.  $D'': 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, |J|=r$  である.

$$\begin{aligned} V &= \iint_{D''} \{3(r \cos \theta + 3)^2 + 2(r \cos \theta + 3)(r \sin \theta + 2)\} r dr d\theta \\ &= \int_0^1 dr \int_0^{2\pi} d\theta (3r^2 \cos^2 \theta + 2r^2 \cos \theta \sin \theta + 22r \cos \theta + 6r \sin \theta + 39)r \end{aligned}$$

ここで  $\theta$  から先に積分すれば

$$\int_0^{2\pi} \cos^2 \theta d\theta = 4 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi, \quad \int_0^{2\pi} \cos \theta \sin \theta d\theta = \int_0^{2\pi} \frac{\sin 2\theta}{2} d\theta = \left[ -\frac{\cos 2\theta}{4} \right]_0^{2\pi} = 0$$

$$, \quad \int_0^{2\pi} \cos \theta d\theta = \int_0^{2\pi} \sin \theta d\theta = 0, \quad \int_0^{2\pi} d\theta = 2\pi \quad \text{であるから,}$$

$$V = \int_0^1 (3r^2 \pi + 78\pi) r dr = \pi \int_0^1 (3r^3 + 78r) dr = \frac{3\pi}{4} + 39\pi = \frac{159}{4} \pi$$

$$V = \frac{159}{4} \pi$$

2.(20 点)

$xy$  平面に対する正射影  $D$  は  $0 \leq z \leq 8, x^2 + y^2 = 8z$  より,  $x^2 + y^2 \leq 64$  である.

極座標変換 ( $x = r \cos \theta, y = r \sin \theta$ ) を使う.

$$z = \frac{1}{8} r^2, z_r = \frac{1}{4} r, z_\theta = 0, D': 0 \leq r \leq 8, 0 \leq \theta \leq 2\pi \text{ である.}$$

$$S = \iint_{D'} \sqrt{r^2 + (rz_r)^2 + (z_\theta)^2} dr d\theta = \int_0^8 dr \int_0^{2\pi} d\theta \sqrt{r^2 + \frac{1}{16} r^4} = 2\pi \int_0^8 r \sqrt{1 + \frac{1}{16} r^2} dr$$

$$1 + \frac{1}{16} r^2 = u \text{ とおく. } du = \frac{1}{8} r dr \text{ である.}$$

$$S = 2\pi \int_1^5 \sqrt{u} 8du = 16\pi \int_1^5 \sqrt{u} du = 16\pi \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^5 = \frac{32}{3} (5\sqrt{5} - 1)\pi$$

$$S = \frac{32}{3} (5\sqrt{5} - 1)\pi$$

3.(20 点)

$S = 2\pi \int_2^{10} y \sqrt{1+(y')^2} dx$  である.  $y^2 = 16x - x^2$  より,  $y = \sqrt{16x - x^2}$ ,  $y' = \frac{16 - 2x}{2\sqrt{16x - x^2}}$  である.

$$S = 2\pi \int_2^{10} \sqrt{16x - x^2} \cdot \sqrt{1 + \frac{(8-x)^2}{16x - x^2}} dx = 2\pi \int_2^{10} \sqrt{64} dx = 128\pi$$

$$S = 128\pi$$

4.(20 点)

幾何学的性質から重心は z 軸上にある.  $\bar{x} = \bar{y} = 0$  である. 球面座標変換を使う.

$$E': 0 \leq r \leq 5, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq 2\pi$$

$$\begin{aligned} M &= \iiint_E \rho dx dy dz = k \iiint_{E'} dx dy dz = k \iiint_{E'} r^2 \sin \theta dr d\theta d\phi \\ &= k \left( \int_0^5 r^2 dr \right) \left( \int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) \left( \int_0^{2\pi} d\phi \right) = k \frac{125}{3} \cdot 1 \cdot 2\pi = \frac{250}{3} \pi k \end{aligned}$$

$$\bar{z} = \frac{1}{M} \iiint_E \rho z dx dy dz$$

$$\begin{aligned} \bar{z} &= \frac{1}{M} \iiint_{E'} kr \cos \theta \cdot r^2 \sin \theta dr d\theta d\phi = \frac{k}{M} \int_0^5 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi r^3 \cos \theta \sin \theta \\ &= \frac{k}{M} \left( \int_0^5 r^3 dr \right) \left( \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta \right) \left( \int_0^{2\pi} d\phi \right) = \frac{2\pi k}{M} \left[ \frac{r^4}{4} \right]_0^5 \left[ \frac{\sin^2 \theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{2\pi k}{M} \cdot \frac{5^4}{4} \cdot \frac{1}{2} = \frac{625\pi k}{4M} = \frac{15}{8} \end{aligned}$$

$$G(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{15}{8})$$

5.(20 点)

球面座標変換より,  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $E': 0 \leq r \leq 4, 0 \leq \theta \leq \pi, 0 \leq \phi \leq \frac{\pi}{2}$  である.

$$I_z = \iiint_E \rho(x, y, z)(x^2 + y^2) dx dy dz$$

$$\begin{aligned} &= k \int_0^4 dr \int_0^\pi d\theta \int_0^{\frac{\pi}{2}} d\phi (r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi) r^2 \sin \theta = k \left( \int_0^4 r^4 dr \right) \left( \int_0^\pi \sin^3 \theta d\theta \right) \left( \int_0^{\frac{\pi}{2}} d\phi \right) \\ &= k \cdot \frac{4^5}{5} \cdot 2 \left( \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \right) \cdot \frac{\pi}{2} = \frac{4^5}{5} \cdot 2 \cdot \frac{2}{3} \cdot \frac{1}{2} \pi k = \frac{2048}{15} \pi k \end{aligned}$$

$$I_z = \frac{2048}{15} \pi k$$