数学 I　レポート No.10　解答　　2017 年度

1.(5 点×4=20 点)

(1)

極座標変換を使う.　$D: x^2+y^2 \le a^2, 0 \le y \le x \to D': 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}$

$$I = \iint_D \sqrt{x^2+y^2}\,dxdy = \iint_{D'} r \cdot r dr d\theta = \int_0^a r^2 dr \int_0^{\frac{\pi}{4}} d\theta = \left[\frac{r^3}{3}\right]_0^a \cdot \frac{\pi}{4} = \frac{\pi}{12}a^3$$

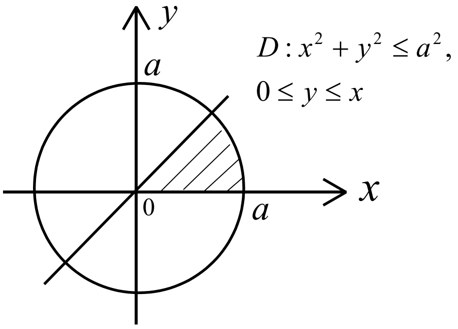

Fig.1-(1)

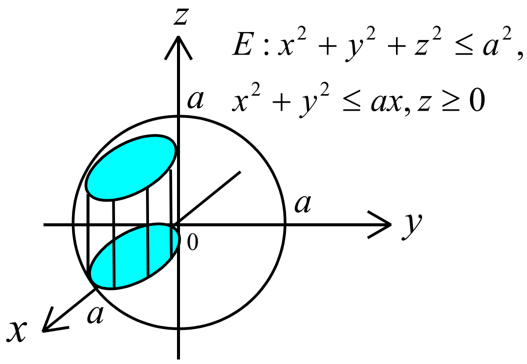

Fig.1-(3)

(2)

球面座標変換を使う.　$E: a^2 \le x^2+y^2+z^2 \le b^2 \to E': a \le r \le b, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$

$$I = \iiint_E \frac{z^2 e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}} dxdydz = \iiint_{E'} \frac{r^2 \cos^2\theta e^{-r^2}}{r} \cdot r^2 \sin\theta dr d\theta d\phi$$

$$= \left(\int_a^b r^3 e^{-r^2} dr\right)\left(\int_0^{\pi} \cos^2\theta \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

ここで $u = r^2$ とおく.　$du = 2rdr, rdr = \frac{du}{2}$

$$I = \left(\int_{a^2}^{b^2} ue^{-u} \frac{du}{2}\right)\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi} \cdot 2\pi = \frac{1}{2}\left[-ue^{-u} - e^{-u}\right]_{a^2}^{b^2} \frac{2}{3} \cdot 2\pi = \frac{2}{3}\pi\{(1+a^2)e^{-a^2} - (1+b^2)e^{-b^2}\}$$

(3)

円柱座標変換を使う.　$x = r\cos\theta, y = r\sin\theta, z = z$

$$E: x^2+y^2+z^2 \le a^2, x^2+y^2 \le ax, z \ge 0 \to E': 0 \le z \le \sqrt{a^2-r^2}, 0 \le r \le a\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$I=\iiint_{E'} r\cos\theta zrdrd\theta dz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{a\cos\theta}dr\cdot r^2\int_0^{\sqrt{a^2-r^2}}zdz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{a\cos\theta}dr\cdot r^2\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-r^2}}$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{a\cos\theta}dr\cdot r^2(a^2-r^2)=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{a\cos\theta}(a^2r^2-r^4)dr=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\left[\frac{a^2r^3}{3}-\frac{r^5}{5}\right]_0^{a\cos\theta}$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{a^5\cos^3\theta}{3}-\frac{a^5\cos^5\theta}{5}\right)\cos\theta d\theta=\frac{a^5}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{3}\cos^4\theta-\frac{1}{5}\cos^6\theta\right)d\theta=a^5\int_0^{\frac{\pi}{2}}\left(\frac{1}{3}\cos^4\theta-\frac{1}{5}\cos^6\theta\right)d\theta$$

$$=a^5\left(\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{5}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)=a^5\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{32}a^5$$

(4)

$x=au, y=bv, z=cw$ とおく．このとき $|J|=abc$ となる．

$$E:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\to E':u^2+v^2+w^2\le 1$$

$$I=\iiint_E x^2y^2z^2dxdydz=\iiint_{E'}a^2u^2b^2v^2c^2w^2abcdudvdw=a^3b^3c^3\iiint_{E'}u^2v^2w^2dudvdw$$

さらに球座標変換を使う． $u=r\sin\theta\cos\phi, v=r\sin\theta\sin\phi, w=r\cos\theta$

$$I=a^3b^3c^3\int_0^1 dr\int_0^{\pi}d\theta\int_0^{2\pi}d\phi(r\sin\theta\cos\phi)^2(r\sin\theta\sin\phi)^2(r\cos\theta)^2(r^2\sin\theta)$$

$$=a^3b^3c^3\int_0^1 r^8dr\int_0^{\pi}\sin^5\theta\cos^2\theta d\theta\int_0^{2\pi}(\cos\phi\sin\phi)^2d\phi=a^3b^3c^3\int_0^1 r^8dr\int_0^{\pi}(\sin^5\theta-\sin^7\theta)d\theta\int_0^{2\pi}\left(\frac{\sin 2\phi}{2}\right)^2d\phi$$

$$=\frac{a^3b^3c^3}{4}\cdot\frac{1}{9}\cdot 2\int_0^{\frac{\pi}{2}}(\sin^5\theta-\sin^7\theta)d\theta\int_0^{4\pi}\sin^2\alpha d\alpha\cdot\frac{1}{2}=\frac{a^3b^3c^3}{36}\cdot\left(\frac{4}{5}\cdot\frac{2}{3}-\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\right)\cdot 8\int_0^{\frac{\pi}{2}}\sin^2\alpha d\alpha$$

$$=a^3b^3c^3\frac{2}{9}\cdot\frac{1}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{4\pi}{945}a^3b^3c^3$$

2.(5 点×2=10 点)

(1)

球面座標変換を使う． $E:x^2+y^2+z^2\le 4\to E':0\le r\le 1,0\le\theta\le\pi,0\le\phi\le 2\pi$

不連続点をなくすため，領域を変更する．

$E_a:x^2+y^2+z^2\le a^2<4\to E'_a:0\le r\le a<2,0\le\theta\le\pi,0\le\phi\le 2\pi$

$$I_a=\iiint_E\frac{1}{\sqrt{1-x^2-y^2-z^2}}dxdydz=\int_0^a dr\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\frac{r^2\sin\theta}{\sqrt{4-r^2}}=\int_0^a\frac{r^2}{\sqrt{4-r^2}}dr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi$$

$$=\int_0^a\frac{r^2}{\sqrt{4-r^2}}dr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi=\left[-\cos\theta\right]_0^{\pi}\cdot 2\pi\int_0^a\frac{r^2}{\sqrt{4-r^2}}dr=4\pi\int_0^a\frac{r^2}{\sqrt{4-r^2}}dr$$

ここで,　$I_1=\int\frac{r^2}{\sqrt{4-r^2}}dr$ とおく.

$$r=2\sin u, dr=2\cos u du, I_1=\int\frac{4\sin^2 u}{2\cos u}2\cos u du=\int 4\sin^2 u du$$

$$I_1=4\int\sin^2 u du=4\int\frac{1-\cos 2u}{2}du=2(u-\frac{\sin 2u}{2})+c=2(u-\sin u\cos u)+c=2(\sin^{-1}\frac{r}{2}-\frac{r}{2}\sqrt{1-\frac{r^2}{4}})+c$$

これより

$$I_a=4\pi\left[2(\sin^{-1}\frac{r}{2}-\frac{r}{2}\sqrt{1-\frac{r^2}{4}})\right]_0^a=8\pi(\sin^{-1}\frac{a}{2}-\frac{a}{2}\sqrt{1-\frac{a^2}{4}})$$,

$$I=\lim_{a\to 2}I_a=\lim_{a\to 2}8\pi(\sin^{-1}\frac{a}{2}-\frac{a}{2}\sqrt{1-\frac{a^2}{4}})=8\pi\sin^{-1}1=8\pi\cdot\frac{\pi}{2}=4\pi^2$$

(2)

有界閉集合にするため,領域を変更する.さらに球面座標変換を使う.

$$E: x\ge 0, y\ge 0, z\ge 0\to E_a: x^2+y^2+z^2\le a^2, x\ge 0, y\ge 0, z\ge 0\to E'_a: 0\le r\le a, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}$$

$$I_a=\iiint_{E_a}\frac{\sqrt{x^2+y^2+z^2}}{(4+x^2+y^2+z^2)^3}dxdydz=\iiint_{E_a}\frac{r\cdot r^2\sin\theta}{(4+r^2)^3}drd\theta d\phi=\{\int_0^a\frac{r^3}{(4+r^2)^3}dr\}(\int_0^{\frac{\pi}{2}}\sin\theta d\theta)(\int_0^{\frac{\pi}{2}}d\phi)$$

ここで $u=4+r^2$ とおく.　$du=2rdr, rdr=\frac{du}{2}$

$$I_a=(\int_4^{4+a^2}\frac{u-4}{u^3}\cdot\frac{du}{2})\left[-\cos\theta\right]_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2}=\frac{\pi}{4}\int_4^{4+a^2}(u^{-2}-4u^{-3})du=\frac{\pi}{4}\left[\frac{u^{-1}}{-2+1}-\frac{4u^{-2}}{-3+1}\right]_4^{4+a^2}$$

$$=\frac{\pi}{4}\{\frac{1}{8}-\frac{1}{4+a^2}+\frac{2}{(4+a^2)^2}\}$$

$$I=\lim_{a\to\infty}I_a=\frac{\pi}{32}$$

3.(5 点×3=15 点)

(1)

$x^2+z^2=2^2$ より,　$z=\sqrt{4-x^2}$　および正射影を $D_1: 0\le y\le x\le 2$ とする.

$$V=16\iint_{D_1}zdxdy=16\iint_{D_1}\sqrt{4-x^2}dxdy=16\int_0^2 dx\int_0^x\sqrt{4-x^2}dy=16\int_0^2 x\sqrt{4-x^2}dx$$

$4-x^2=u$ とおく． $-2xdx=du, xdx=-\dfrac{du}{2}$

$$V=16\int_4^0\sqrt{u}(-\frac{du}{2})=8\int_0^4\sqrt{u}du=8\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4=\frac{16}{3}\cdot 8=\frac{128}{3}$$

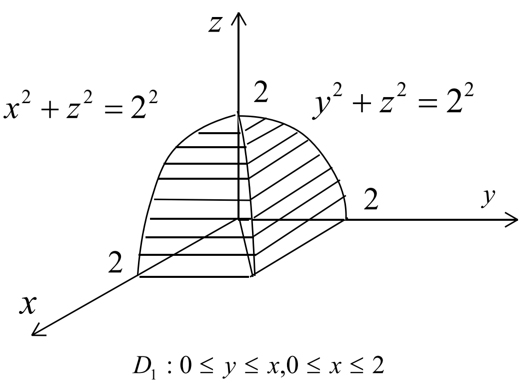

Fig.3-(1)

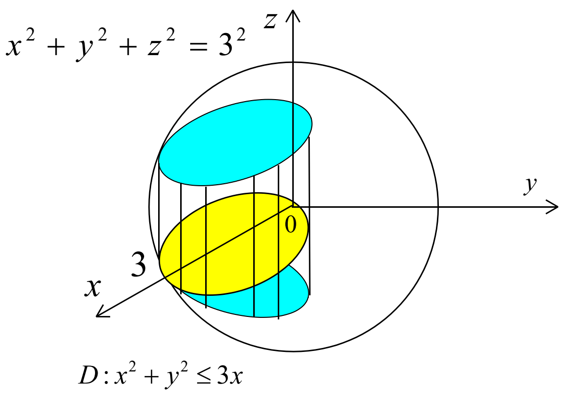

Fig.3-(2)

(2)

xy 平面の正射影は $x^2+y^2=3x$ である． $x^2+y^2+z^2=9$ より $z=\sqrt{9-x^2-y^2}$ になる．

$$V=2\iint_D zdxdy=2\iint_D\sqrt{9-x^2-y^2}dxdy$$

極座標変換を使う． $D: x^2+y^2\le 3x \to D': 0\le r\le 3\cos\theta, -\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$

$V=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{3\cos\theta}r\sqrt{9-r^2}dr$ ここで $9-r^2=u$ とおく． $-2rdr=du, rdr=-\dfrac{du}{2}$ および

$0\le r\le 3\cos\theta \to 9\le u\le 9\sin^2\theta$ となる．

$$V=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_9^{9\sin^2\theta}\sqrt{u}(-\frac{du}{2})=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{9\sin^2\theta}^9\sqrt{u}du=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{9\sin^2\theta}^9$$

$$=\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{(9)^{\frac{3}{2}}-(9\sin^2\theta)^{\frac{3}{2}}\}d\theta=\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}27(1-|\sin\theta|^3)d\theta=36\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta$$

$$=36(\int_0^{\frac{\pi}{2}}d\theta-\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta)=36(\frac{\pi}{2}-\frac{2}{3})=18\pi-24$$

(3)

$E: 0 \le z \le \frac{1}{6}(x^2+y^2), x^2+y^2 \le 2x$ の体積である．

xy 平面の正射影は $D: x^2+y^2 \le 2x$ である．また $0 \le z \le \frac{1}{6}(x^2+y^2)$ である．

$V = \iiint_E dxdydz = \iint_D dxdy \int_0^{\frac{1}{6}(x^2+y^2)} dz = \iint_D \frac{1}{6}(x^2+y^2)dxdy$　　極座標変換を使う．

$D: x^2+y^2 \le 2x \to D': 0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ となる．

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \frac{1}{6} r^2 \cdot rdr = \frac{1}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3 dr = \frac{1}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta} = \frac{2^4}{24}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

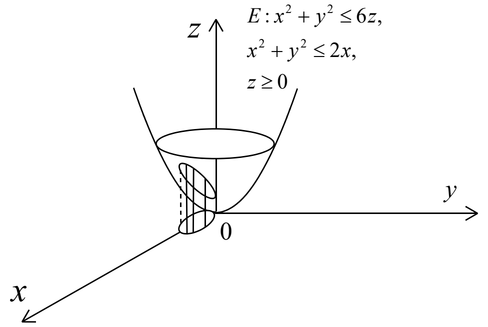

Fig.3-(3)

4.(5 点×3=15 点)

(1)

$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ より， $z = 4(1-\frac{x}{2}-\frac{y}{3}), z_x = -2, z_y = -\frac{4}{3}$ である． xy 平面に対する正射影は

$z=0$ とおいて， $\frac{x}{2}+\frac{y}{3} \le 1$ となる．これより $D: 0 \le y \le 3(1-\frac{x}{2}), 0 \le x \le 2$ である．

$$S = \iint_D \sqrt{1+z_x^2+z_y^2}dxdy = \int_0^2 dx \int_0^{3(1-\frac{x}{2})} dy \sqrt{1+4+\frac{16}{9}} = \frac{\sqrt{61}}{3}\int_0^2 3(1-\frac{x}{2})dx$$

$$= \sqrt{61}\left[x-\frac{x^2}{4}\right]_0^2 = \sqrt{61}$$

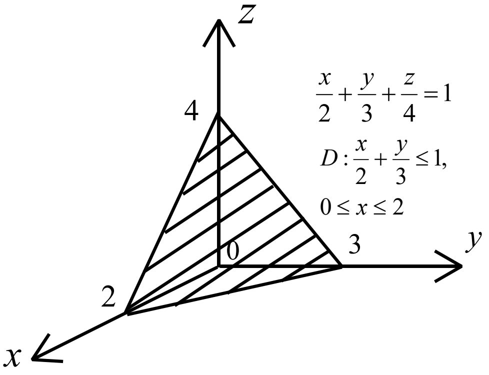

Fig.4-(1)

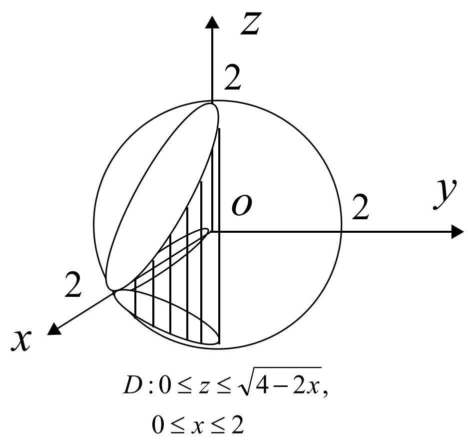

Fig.4-(2)

(2)

xz 面に対する正射影は， $x^2+y^2+z^2=2^2, x^2+y^2=2x$ から $z^2+2x=4$ である．

これより， $z=\pm\sqrt{4-2x}$ また $y^2=2x-x^2\ge 0$ から $0\le x\le 2$ となる． $y=\pm\sqrt{2x-x^2}$

求める面積は $S_1(y\ge 0, z\ge 0)$ の 4 倍になる． $y=\sqrt{2x-x^2}, y_x=\dfrac{2-2x}{2\sqrt{2x-x^2}}, y_z=0$ である．

$$S=4S_1=4\iint_{D_1}\sqrt{1+y_x^2+y_z^2}\,dxdz=4\int_0^2 dx\int_0^{\sqrt{4-2x}}\sqrt{1+\frac{(2-2x)^2}{4(2x-x^2)}}dz=4\int_0^2 dx\int_0^{\sqrt{4-2x}}\sqrt{\frac{1}{2x-x^2}}dz$$

$$=4\int_0^2 dx\left[\frac{z}{\sqrt{2x-x^2}}\right]_0^{\sqrt{4-2x}}=4\int_0^2\sqrt{\frac{2(2-x)}{x(2-x)}}dx=4\sqrt{2}\int_0^2\frac{dx}{\sqrt{x}}=4\sqrt{2}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^2=16$$

(3)

$y=2x^3, y'=6x^2$ より $S=2\pi\int_0^a y\sqrt{1+y'^2}dx=2\pi\int_0^a 2x^3\sqrt{1+36x^4}dx$ 　 $u=1+36x^4$ とおく．

$du=144x^3dx$ である． $0\le x\le a\rightarrow 1\le u\le 1+36a^4$

$$S=2\pi\int_1^{1+36a^4}\sqrt{u}\cdot\frac{du}{72}=\frac{\pi}{36}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{1+36a^4}=\frac{\pi}{54}\{(1+36a^4)^{\frac{3}{2}}-1\}$$

5.(10 点×2=20 点)

(1)

$E:\dfrac{x^2}{16}+\dfrac{y^2}{9}+\dfrac{z^2}{4}\le 1, x\ge 0, y\ge 0, z\ge 0$　$\rho = k$ (一定)とおく． $x=4u, y=3v, z=2w$ と変換する．このとき $|J|=24$ である．さらに球面座標変換を使う．

$$E':u^2+v^2+w^2=1, u\ge 0, v\ge 0, w\ge 0 \to E'':0\le r\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}$$

$$M=\iiint_E \rho dxdydz = k\iiint_{E'} 24dudvdw = 24k\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} d\phi r^2\sin\theta = 24k(\int_0^1 r^2dr)(\int_0^{\frac{\pi}{2}}\sin\theta d\theta)(\int_0^{\frac{\pi}{2}} d\phi)$$

$$=24k\cdot\frac{1}{3}\cdot 1\cdot\frac{\pi}{2}=4\pi k$$

$$\bar{x}=\frac{1}{M}\iiint_E \rho x dxdydz=\frac{k}{M}\iiint_E xdxdydz=\frac{k}{M}\iiint_{E'} 24\cdot 4ududvdw$$

$$=\frac{96k}{M}\iiint_{E''} r\sin\theta\cos\phi\cdot r^2\sin\theta drd\theta d\phi=\frac{96k}{M}(\int_0^1 r^3dr)(\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta)(\int_0^{\frac{\pi}{2}}\cos\phi d\phi)$$

$$=\frac{96k}{M}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot 1=\frac{96k\pi}{16M}=\frac{96k\pi}{16\cdot 4k\pi}=\frac{3}{2}$$

$$\bar{y}=\frac{1}{M}\iiint_E \rho y dxdydz=\frac{k}{M}\iiint_E ydxdydz=\frac{k}{M}\iiint_{E'} 72vdudvdw$$

$$=\frac{72k}{M}\iiint_{E''} r\sin\theta\sin\phi\cdot r^2\sin\theta drd\theta d\phi=\frac{72k}{M}(\int_0^1 r^3dr)(\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta)(\int_0^{\frac{\pi}{2}}\sin\phi d\phi)$$

$$=\frac{72k}{M}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot 1=\frac{72k\pi}{16M}=\frac{72k\pi}{16\cdot 4kk}=\frac{9}{8}$$

$$\bar{z}=\frac{1}{M}\iiint_E \rho z dxdydz=\frac{k}{M}\iiint_E zdxdydz=\frac{k}{M}\iiint_{E'} 48wdudvdw$$

$$=\frac{48k}{M}\iiint_{E''} r\cos\theta\cdot r^2\sin\theta drd\theta d\phi=\frac{48k}{M}(\int_0^1 r^3dr)(\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta)(\int_0^{\frac{\pi}{2}} d\phi)$$

$$=\frac{48k}{M}\cdot\frac{1}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta\cdot\frac{\pi}{2}=\frac{48k\pi}{16M}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{3k\pi}{M}=\frac{3k\pi}{4k\pi}=\frac{3}{4}$$

$$G(\bar{x},\bar{y},\bar{z})=(\frac{3}{2},\frac{9}{8},\frac{3}{4})$$

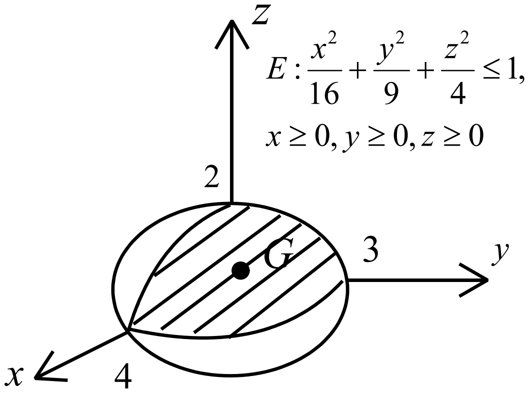

Fig.5-(1)

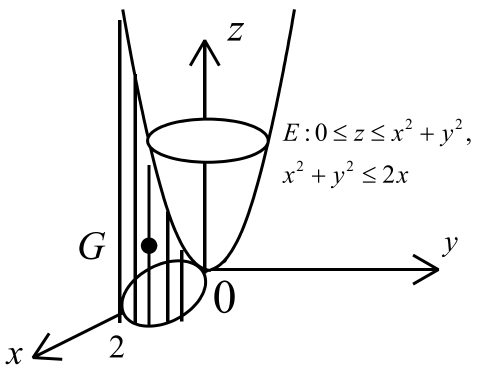

Fig.5-(2)

(2)

$\rho = k$ (一定)とおく.

$E: 0 \le z \le x^2 + y^2, x^2 + y^2 \le 2x$ 質量を M とする.

$M = \iiint_E \rho dxdydz = k\iint_D dxdy\int_0^{x^2+y^2} dz = k\iint_D dxdy(x^2+y^2)$ ここで正射影 D に対し極座標変換を使う.

$D: x^2 + y^2 \le 2x \to D': 0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$M = k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^2 \cdot rdr = k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta} = k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^4\theta d\theta = 8k\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= 8k \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{2}k\pi$$

$$\bar{x} = \frac{1}{M}\iiint_E \rho xdxdydz = \frac{k}{M}\iiint_E xdxdydz = \frac{k}{M}\iint_D xdxdy\int_0^{x^2+y^2} dz = \frac{k}{M}\iint_D x(x^2+y^2)dxdy$$

$$= \frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r\cos\theta \cdot r^2 \cdot rdr = \frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos\theta \int_0^{2\cos\theta} r^4 dr = \frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos\theta \left[\frac{r^5}{5}\right]_0^{2\cos\theta}$$

$$= \frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{32}{5}\cos^6\theta d\theta = \frac{64k}{5M}\int_0^{\frac{\pi}{2}} \cos^6\theta d\theta = \frac{64k}{5M} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi k}{16M} = \frac{4}{3}$$

$$\bar{y} = \frac{1}{M}\iiint_E \rho ydxdydz = \frac{k}{M}\iiint_E ydxdydz = \frac{k}{M}\iint_D ydxdy\int_0^{x^2+y^2} dz = \frac{k}{M}\iint_D y(x^2+y^2)dxdy$$

$$=\frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r\sin\theta\cdot r^2\cdot rdr=\frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\sin\theta\int_0^{2\cos\theta}r^4dr=\frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\sin\theta\left[\frac{r^5}{5}\right]_0^{2\cos\theta}$$

$$=\frac{k}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{32}{5}\sin\theta\cos^5\theta d\theta=\frac{32k}{5M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cos^5\theta d\theta=0$$

$$\bar{z}=\frac{1}{M}\iiint_E\rho zdxdydz=\frac{k}{M}\iiint_E zdxdydz=\frac{k}{M}\iint_D dxdy\int_0^{x^2+y^2}zdz=\frac{k}{M}\iint_D dxdy\left[\frac{z^2}{2}\right]_0^{x^2+y^2}$$

$$=\frac{k}{M}\iint_D\frac{(x^2+y^2)^2}{2}dxdy=\frac{k}{2M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^4\cdot rdr=\frac{k}{2M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^6}{6}\right]_0^{2\cos\theta}d\theta=\frac{64k}{12M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^6\theta d\theta$$

$$=\frac{k64\cdot 2}{12\cdot\frac{3}{2}k\pi}\int_0^{\frac{\pi}{2}}\cos^6\theta d\theta=\frac{128}{18\pi}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{10}{9}\qquad G(\bar{x},\bar{y},\bar{z})=(\frac{4}{3},0,\frac{10}{9})$$

6.(10 点×2＝20 点)

(1)

球面座標変換を使う.

$$E:x^2+y^2+z^2\leq 4, z\geq 0\rightarrow E':0\leq r\leq 2, 0\leq\theta\leq\frac{\pi}{2}, 0\leq\phi\leq 2\pi$$

$$I_z=\iiint_E\rho(x^2+y^2)dxdydz=k\int_0^2dr\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\pi}d\phi\{(r\sin\theta\cos\phi)^2+(r\sin\theta\sin\phi)^2\}r^2\sin\theta$$

$$=k(\int_0^2r^4dr)(\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta)(\int_0^{2\pi}d\phi)=k\cdot\frac{32}{5}\cdot\frac{2}{3}\cdot 2\pi=\frac{128\pi k}{15}$$

$$M=\frac{4}{3}\pi 2^3\cdot\frac{1}{2}\cdot k=\frac{16\pi k}{3}$$

$$I_l=I_z+Mb^2=\frac{128\pi k}{15}+\frac{16\pi k\cdot 3^2}{3}=\pi k(\frac{128}{15}+48)=\frac{848}{15}\pi k$$

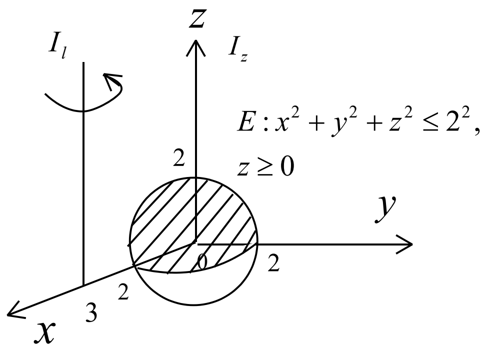

Fig.6-(1)

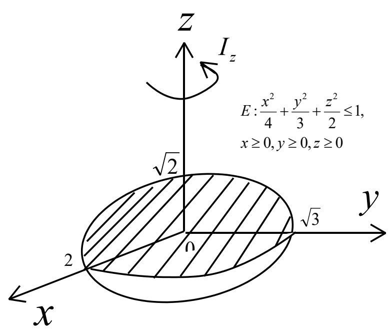

Fig.6-(2)

(2)

$x = 2u, y = \sqrt{3}v, z = \sqrt{2}w$ とした後，球面座標変換を使う．

$$E: \frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{2} \le 1, x \ge 0, y \ge 0, z \ge 0 \to E': u^2+v^2+w^2 \le 1, u \ge 0, v \ge 0, w \ge 0$$

$$\to E'': 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}$$

$$I_z = \iiint_E \rho(x^2+y^2)dxdydz = \iiint_{E'} \rho(4u^2+3v^2)2\sqrt{6}dudvdw$$

$$= k2\sqrt{6}\int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\phi \{4(r\sin\theta\cos\phi)^2 + 3(r\sin\theta\sin\phi)^2\} r^2 \sin\theta$$

$$= 2\sqrt{6}k(\int_0^1 r^4 dr)(\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta)\{\int_0^{\frac{\pi}{2}} (4\cos^2\phi + 3\sin^2\phi)d\phi\}$$

$$= 2\sqrt{6}k \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot (4\frac{1}{2} \cdot \frac{\pi}{2} + 3\frac{1}{2} \cdot \frac{\pi}{2}) = \frac{7\sqrt{6}}{15}\pi k$$