1.(10 点×5)

(1)

$$I=\int_0^1 dy\int_0^1(3x^2y^3+5x^3y^2)dx=\int_0^1 dy\left[x^3y^3+\frac{5x^4y^2}{4}\right]_0^1=\int_0^1(y^3+\frac{5y^2}{4})dy=\left[\frac{y^4}{4}+\frac{5y^3}{12}\right]_0^1=\frac{2}{3}$$

(2)

$$I=\int_0^1 dy\int_0^1\frac{1}{(1+3x)(1+4y)}dx=\int_0^1 dy\frac{1}{1+4y}\left[\frac{1}{3}\log(1+3x)\right]_0^1=\frac{\log 4}{3}\int_0^1\frac{1}{1+4y}dy$$

$$=\frac{\log 4}{3}\left[\frac{1}{4}\log(1+4y)\right]_0^1=\frac{\log 4\log 5}{12}=\frac{\log 2\log 5}{6}$$

(3)

$$I=\int_0^{\frac{\pi}{2}}dy\int_0^{\frac{\pi}{2}}\sin(2x+3y)dx=\int_0^{\frac{\pi}{2}}dy\left[-\frac{1}{2}\cos(2x+3y)\right]_0^{\frac{\pi}{2}}=\frac{1}{2}\int_0^{\frac{\pi}{2}}\{-\cos(\pi+3y)+\cos 3y\}dy$$

$$=\frac{1}{6}[-\sin(\pi+3y)+\sin 3y]_0^{\frac{\pi}{2}}=\frac{1}{6}(-\sin\frac{5}{2}\pi+\sin\frac{3}{2}\pi+\sin\pi-0)=-\frac{1}{3}$$

(4)

$$I=\int_0^1 dy\int_0^1\frac{xy^3}{1+5x^2y^2}dx$$ ，$5x^2y^2=t$ とおく．$5x^2y^2=t$,$10xy^2dx=dt, xdx=\dfrac{dt}{10y^2}$

$$I=\int_0^1 dy\int_0^{5y^2}\frac{y^3\times\dfrac{dt}{10y^2}}{1+t}=\frac{1}{10}\int_0^1 ydy\int_0^{5y^2}\frac{1}{1+t}dt=\frac{1}{10}\int_0^1 ydy[\log(1+t)]_0^{5y^2}=\frac{1}{10}\int_0^1 y\log(1+5y^2)dy$$

$1+5y^2=z$ とおく．$10ydy=dz$

$$I=\frac{1}{10}\int_1^6\log z\frac{dz}{10}=\frac{1}{100}[z\log z-z]_1^6=\frac{1}{100}(6\log 6-6+1)=\frac{1}{100}(6\log 6-5)$$

(5)

$$I=\int_0^1 dy\int_0^1 y\log(1+4xy)dx$$ 　$1+4xy=t$ とおく．$4ydx=dt$

$$I=\int_0^1 dy\int_1^{1+4y}y\log t\frac{dt}{4y}=\frac{1}{4}\int_0^1 dy\int_1^{1+4y}\log tdt=\frac{1}{4}\int_0^1 dy[t\log t-t]_1^{1+4y}$$

$$=\frac{1}{4}\int_0^1[\{(1+4y)\log(1+4y)-(1+4y)\}-(\log 1-1)]dy=\frac{1}{4}\int_0^1\{(1+4y)\log(1+4y)-4y\}dy$$

$1+4y=z$ とおく.

$I=\frac{1}{16}\int_1^5 z\log z dz-\int_0^1 y dy$ ここで $u'=z, v=\log z$ とおく。$u=\frac{z^2}{2}, v'=\frac{1}{z}$

$J=\int z\log z dz=\frac{z^2}{2}\log z-\int\frac{z}{2}dz=\frac{1}{2}z^2\log z-\frac{1}{4}z^2+c$

$I=\frac{1}{16}\left[\frac{1}{2}z^2\log z-\frac{1}{4}z^2\right]_1^5-\frac{1}{2}=\frac{25}{32}\log 5-\frac{7}{8}$

2.(10 点×5)

(1)

$I=\int_0^1 dz\int_0^1 dy\int_0^1 (x^3+2y^3+3z^3+xyz)dx=\int_0^1 dz\int_0^1 dy\left[\frac{x^4}{4}+2y^3x+3z^3x+\frac{1}{2}x^2yz\right]_0^1$

$=\int_0^1 dz\int_0^1 dy(\frac{1}{4}+2y^3+3z^3+\frac{1}{2}yz)=\int_0^1 dz\left[\frac{1}{4}y+\frac{2}{4}y^4+3z^3y+\frac{1}{4}y^2z\right]_0^1=\int_0^1(\frac{3}{4}+3z^3+\frac{1}{4}z)dz$

$=\left[\frac{3z}{4}+\frac{3z^4}{4}+\frac{1}{8}z^2\right]_0^1=\frac{3}{4}+\frac{3}{4}+\frac{1}{8}=\frac{13}{8}$

(2)

$I=\int_0^{\frac{\pi}{4}} dz\int_0^{\frac{\pi}{4}} dy\int_0^{\frac{\pi}{4}}\sin(3x+y+2z)dx=\int_0^{\frac{\pi}{4}} dz\int_0^{\frac{\pi}{4}} dy\left[-\frac{1}{3}\cos(3x+y+2z)\right]_0^{\frac{\pi}{4}}$

$=\int_0^{\frac{\pi}{4}} dz\int_0^{\frac{\pi}{4}} dy\frac{1}{3}\{-\cos(\frac{3\pi}{4}+y+2z)+\cos(y+2z)\}=\int_0^{\frac{\pi}{4}} dz\frac{1}{3}\left[-\sin(\frac{3\pi}{4}+y+2z)+\sin(y+2z)\right]_0^{\frac{\pi}{4}}$

$=\int_0^{\frac{\pi}{4}} dz\frac{1}{3}\{-\sin(\frac{3\pi}{4}+\frac{\pi}{4}+2z)+\sin(\frac{\pi}{4}+2z)+\sin(\frac{3\pi}{4}+2z)-\sin(2z)\}$

$=\frac{1}{6}\left[\cos(\frac{3\pi}{4}+\frac{\pi}{4}+2z)-\cos(\frac{\pi}{4}+2z)-\cos(\frac{3\pi}{4}+2z)+\cos(2z)\right]_0^{\frac{\pi}{4}}$

$=\frac{1}{6}\{\cos(\pi+\frac{\pi}{2})-\cos(\frac{\pi}{4}+\frac{\pi}{2})-\cos(\frac{3\pi}{4}+\frac{\pi}{2})+\cos(\frac{\pi}{2})-\cos(\pi)+\cos(\frac{\pi}{4})+\cos(\frac{3\pi}{4})-\cos(0)\}$

$=\frac{1}{6}(0+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-1)=\frac{\sqrt{2}}{6}$

(3)

$I=\int_1^2 dz\int_1^2 dy\int_1^2\frac{x^2+3y^2+2z^2}{xyz}dx=\int_1^2 dz\int_1^2 dy\int_1^2(\frac{x}{yz}+\frac{3y}{xz}+\frac{2z}{xy})dx$

$=\int_1^2 dz\int_1^2 dy\left[\frac{x^2}{2yz}+\frac{3y}{z}\log x+\frac{2z}{y}\log x\right]_1^2=\int_1^2 dz\int_1^2 dy(\frac{2}{yz}+\frac{3y}{z}\log 2+\frac{2z}{y}\log 2-\frac{1}{2yz})$

$$=\int_1^2 dz\left[\frac{2}{z}\log y+\frac{3\log 2}{2z}y^2+(2\log 2)z\log y-\frac{1}{2z}\log y\right]_1^2$$

$$=\int_1^2 dz\{\frac{2\log 2}{z}+\frac{6\log 2}{z}+2(\log 2)^2 z-\frac{\log 2}{2z}-\frac{3\log 2}{2z}\}=\int_1^2\{\frac{6\log 2}{z}+2(\log 2)^2 z\}dz$$

$$=\left[(6\log 2)\log z+(\log 2)^2 z^2\right]_1^2=6(\log 2)^2+4(\log 2)^2-(\log 2)^2=9(\log 2)^2$$

(4)

$$I=\int_0^1 dz\int_0^1 dy\int_0^1\frac{dx}{1+x+y+z}=\int_0^1 dz\int_0^1 dy\left[\log(1+x+y+z)\right]_0^1=\int_0^1 dz\int_0^1\{\log(2+y+z)-\log(1+y+z)\}dy$$

ここで$\int\log x dx=x\log x-x+c$である.

$$I=\int_0^1 dz\{[(2+y+z)\log(2+y+z)-(2+y+z)]_0^1-[(1+y+z)\log(1+y+z)-(1+y+z)]_0^1\}$$

$$=\int_0^1 dz[\{(3+z)\log(3+z)-(3+z)\}-\{(2+z)\log(2+z)-(2+z)\}-\{(2+z)\log(2+z)-(2+z)\}$$
$$+\{(1+z)\log(1+z)-(1+z)\}]$$
$$=\int_0^1\{(3+z)\log(3+z)-2(2+z)\log(2+z)+(1+z)\log(1+z)\}dz$$

ここで$J=\int x\log x dx$， $u'=x, v=\log x$とおく． $u=\frac{x^2}{2}, v'=\frac{1}{x}$より，

$J=\int x\log x dx=\frac{x^2}{2}\log x-\int\frac{x}{2}dx=\frac{x^2}{2}\log x-\frac{x^2}{4}+c$である．

$$I=\left[\{\frac{(3+z)^2}{2}\log(3+z)-\frac{(3+z)^2}{4}\}-2\{\frac{(2+z)^2}{2}\log(2+z)-\frac{(2+z)^2}{4}\}+\{\frac{(1+z)^2}{2}\log(1+z)-\frac{(1+z)^2}{4}\}\right]_0^1$$

$$=\{(\frac{4^2}{2}\log 4-\frac{4^2}{4})-2(\frac{3^2}{2}\log 3-\frac{3^2}{4})+(\frac{2^2}{2}\log 2-\frac{2^2}{4})\}-\{(\frac{3^2}{2}\log 3-\frac{3^2}{4})-2(\frac{2^2}{2}\log 2-\frac{2^2}{4})+(\frac{1^2}{2}\log 1-\frac{1^2}{4})\}$$
$$=\{(16\log 2-4)-2(\frac{9}{2}\log 3-\frac{9}{4})+(2\log 2-1)\}-\{(\frac{9}{2}\log 3-\frac{9}{4})-2(2\log 2-1)-\frac{1}{4}\}$$
$$=16\log 2-4-9\log 3+\frac{9}{2}+2\log 2-1-\frac{9}{2}\log 3+\frac{9}{4}+4\log 2-2+\frac{1}{4}=22\log 2-\frac{27}{2}\log 3-7+\frac{28}{4}$$
$$=22\log 2-\frac{27}{2}\log 3$$

(5)

$$I=\int_0^1 dz\int_0^1 dy\int_0^1\sqrt{x+y+z}\,dx=\int_0^1 dz\int_0^1 dy\left[\frac{(x+y+z)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1=\int_0^1 dz\int_0^1\frac{2}{3}\{(1+y+z)^{\frac{3}{2}}-(y+z)^{\frac{3}{2}}\}dy$$

$$=\int_0^1 dz\,\frac{2}{3}\left[\frac{(1+y+z)^{\frac{5}{2}}-(y+z)^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1=\int_0^1\frac{4}{15}\{(z+2)^{\frac{5}{2}}-(z+1)^{\frac{5}{2}}-(z+1)^{\frac{5}{2}}+z^{\frac{5}{2}}\}dz$$

$$=\frac{4}{15}\left[\frac{(z+2)^{\frac{7}{2}}-2(z+1)^{\frac{7}{2}}+z^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1=\frac{8}{105}(3^{\frac{7}{2}}-2\cdot 2^{\frac{7}{2}}+1-2^{\frac{7}{2}}+2)=\frac{8}{105}(27\sqrt{3}-24\sqrt{2}+3)$$

$$=\frac{8}{35}(9\sqrt{3}-8\sqrt{2}+1)$$