数学Ｉ　レポート No.7　解答　2017 年度

1.(4 点×3)

(1)

$\iint_D f(x,y)dxdy = \iint_{D'} f(r\cos\theta, r\sin\theta) rdrd\theta$ より　① $= rdrd\theta$

(2)

$\iiint_D f(x,y,z)dxdydz = \iiint_{D'} (r\cos\theta, r\sin\theta, z) rdrd\theta dz$ より　② $= rdrd\theta dz$

(3)

$\iiint_D f(x,y,z)dxdydz = \iiint_{D'} (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2\sin\theta drd\theta d\phi$ より

③ $= r^2\sin\theta drd\theta d\phi$

2.(8 点×2)

(1)

$$I = \iint_D xy\sqrt{x^2+y^2}dxdy = \iint_{D'} r\cos\theta r\sin\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, rdrd\theta = \iint_{D'} r^4\frac{1}{2}\sin 2\theta drd\theta$$

$$= (\int_1^2 r^4 dr)(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta) = \left[\frac{r^5}{5}\right]_1^2\left[-\frac{\cos 2\theta}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{31}{20}$$

$1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

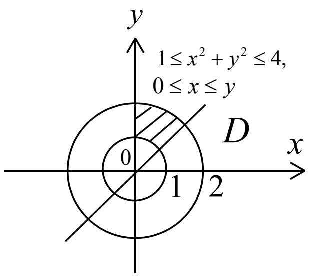


Fig.2-(1)

(2)

$$I = \iint_D \sin\{\frac{\pi(\sqrt{x^2+y^2})}{4}\}dxdy = \iint_{D'} \sin\frac{\pi r}{4} rdrd\theta = (\int_0^1 r\sin\frac{\pi r}{4}dr)(\int_0^{2\pi} d\theta)$$

, $z = \frac{\pi r}{4}$ とおく． $dr = \frac{4}{\pi}dz$.

$$I=\int_0^{\frac{\pi}{4}}\frac{4}{\pi}z\sin z\cdot\frac{4}{\pi}dz\cdot 2\pi=\frac{32}{\pi}\int_0^{\frac{\pi}{4}}z\sin zdz=\frac{32}{\pi}\left[-z\cos z+\int\cos zdz\right]_0^{\frac{\pi}{4}}=\frac{32}{\pi}\left[-z\cos z+\sin z\right]_0^{\frac{\pi}{4}}$$

$$=\frac{32}{\pi}(-\frac{\pi}{4}\cos\frac{\pi}{4}+\sin\frac{\pi}{4})=\frac{32}{\pi}\frac{\sqrt{2}}{2}(1-\frac{\pi}{4})=\frac{16\sqrt{2}}{\pi}(1-\frac{\pi}{4})$$

3.(8 点×2)

(1)

$$I=\iiint_D\frac{z}{\sqrt{x^2+y^2+z^2}}dxdydz \quad x=r\cos\theta, y=r\sin\theta, z=z \text{ とおく.}$$

$$I=\iiint_{D'}\frac{z}{\sqrt{r^2+z^2}}rdrd\theta dz \quad D':1\le r\le 4, 0\le\theta\le 2\pi, 0\le z\le 2 \text{ である.}$$

$I=\int_0^{2\pi}d\theta\int_0^2 dz\int_1^4\frac{rz}{\sqrt{r^2+z^2}}dr$　$r^2+z^2=u$ とおく． $2rdr=du$ である． $1\le r\le 4$ のとき $1+z^2\le u\le 16+z^2$ になる． $I=\int_0^{2\pi}d\theta\int_0^2 dz\int_{1+z^2}^{16+z^2}\frac{z}{\sqrt{u}}\cdot\frac{du}{2}=2\pi\int_0^2\frac{z}{2}dz\int_{1+z^2}^{16+z^2}u^{-\frac{1}{2}}du$

$I=\pi\int_0^2 zdz\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_{1+z^2}^{16+z^2}=2\pi\int_0^2 z(\sqrt{z^2+16}-\sqrt{z^2+1})dz$,　$I_1=\int z\sqrt{z^2+a^2}dz$ とする.

$v=z^2+a^2$ とおく． $dv=2zdz$ である． $I_1=\int\frac{\sqrt{v}}{2}dv=\frac{1}{2}\cdot\frac{v^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{v^{\frac{3}{2}}}{3}+c$

$I_1=\frac{1}{3}(z^2+a^2)^{\frac{3}{2}}+c$ 故に $I=2\pi\left[\frac{1}{3}(z^2+16)^{\frac{3}{2}}-\frac{1}{3}(z^2+1)^{\frac{3}{2}}\right]_0^2=\frac{14\pi}{3}(5\sqrt{5}-9)$

(2)

$$I=\iiint_D z\sin\{\frac{\pi(x^2+y^2+z^2)}{4}\}dxdydz \quad x=r\ \cos\theta, y=r\ \sin\theta, z=z \quad \text{とおく.}$$

$$D:0\le x^2+y^2\le 1, 0\le z\le 1\to D':0\le r\le 1, 0\le z\le 1, 0\le\theta\le 2\pi$$

$$I=\iiint_{D'}z\sin\{\frac{\pi(r^2+z^2)}{4}\}rdrd\theta dz=\int_0^{2\pi}d\theta\int_0^1 dz\int_0^1 zr\sin\{\frac{\pi(r^2+z^2)}{4}\}dr$$

$I=2\pi\int_0^1 dz\int_0^1 zr\sin\{\frac{\pi(r^2+z^2)}{4}\}dr$ ここで $u=\frac{\pi(r^2+z^2)}{4}$ とおく． $du=\frac{\pi}{2}rdr, rdr=\frac{2du}{\pi}$

$0 \le r \le 1 \to \frac{\pi}{4}z^2 \le u \le \frac{\pi(1+z^2)}{4}$ より $I = 2\pi\int_0^1 dz \int_{\frac{\pi z^2}{4}}^{\frac{\pi(1+z^2)}{4}} z\sin u \frac{2du}{\pi}$

$I = 4\int_0^1 zdz\left[-\cos u\right]_{\frac{\pi z^2}{4}}^{\frac{\pi(1+z^2)}{4}} = 4\int_0^1 \{z\cos\frac{\pi z^2}{4} - z\cos\frac{\pi(1+z^2)}{4}\}dz,\ \ I_1 = \int z\cos\frac{\pi z^2}{4}dz, u = \frac{\pi z^2}{4}$

とおく.

$I_1 = \int \frac{2}{\pi}\cos u du = \frac{2\sin u}{\pi} + c_1 = \frac{2}{\pi}\sin\frac{\pi z^2}{4} + c_1$

$I_2 = \int z\cos\frac{\pi(1+z^2)}{4}dz$ 　とおく. $v = \frac{\pi(1+z^2)}{4}, dv = \frac{\pi}{2}zdz$

$I_2 = \int \cos v\frac{2dv}{\pi} = \frac{2}{\pi}\sin v + c_2 = \frac{2}{\pi}\sin\frac{\pi(1+z^2)}{4} + c_2$

$I = 4\left[\frac{2}{\pi}\sin\frac{\pi z^2}{4} - \frac{2}{\pi}\sin\frac{\pi(1+z^2)}{4}\right]_0^1 = \frac{8}{\pi}\{(\sin\frac{\pi}{4} - \sin\frac{\pi}{2}) - (\sin 0 - \sin\frac{\pi}{4})\} = \frac{8}{\pi}(\sqrt{2}-1)$

4.(8 点×2)

(1)

$I = \iiint_D xe^{x^2+y^2+z^2}dxdydz$ 　　　$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ により

$D: 0 \le x^2 + y^2 + z^2 \le 1, x \ge 0, y \ge 0, z \ge 0 \to D': 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}$ となる.

$I = \iiint_{D'} r\sin\theta\cos\phi e^{r^2} r^2\sin\theta drd\theta d\phi = \int_0^{\frac{\pi}{2}}\cos\phi d\phi\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta\int_0^1 r^3e^{r^2}dr$

$= \left[\sin\phi\right]_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}d\theta\int_0^1 r^3e^{r^2}dr = \left[\frac{\theta - \frac{\sin 2\theta}{2}}{2}\right]_0^{\frac{\pi}{2}}\int_0^1 r^3e^{r^2}dr = \frac{1}{2}\cdot\frac{\pi}{2}\int_0^1 r^3e^{r^2}dr, u = r^2, rdr = \frac{du}{2}$

$I = \frac{\pi}{4}\int_0^1 ue^u\frac{du}{2} = \frac{\pi}{8}\int_0^1 ue^u du = \frac{\pi}{8}\{\left[ue^u\right]_0^1 - \int_0^1 e^u du\} = \frac{\pi}{8}\left[ue^u - e^u\right]_0^1 = \frac{\pi}{8}$

(2)

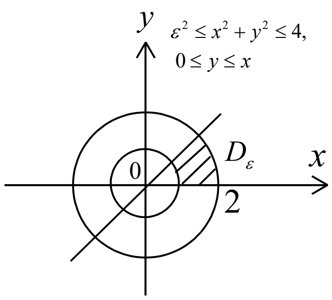
$I = \iiint_D \frac{z}{\sqrt{x^2+y^2+z^2}}dxdydz$ 　　　$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ により

$D: 1 \le x^2 + y^2 + z^2 \le 16, x \ge 0, y \ge 0, z \ge 0 \to D': 1 \le r \le 4, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}$ となる.

$$I=\iiint_{D'}\frac{1}{r}r^3\cos\theta\sin\theta drd\theta d\phi=(\int_1^4 r^2dr)(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta)(\int_0^{\frac{\pi}{2}}d\phi)=\left[\frac{r^3}{3}\right]_1^4\left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2}$$

$$=\frac{63}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{21}{4}\pi$$

5(10 点×2)



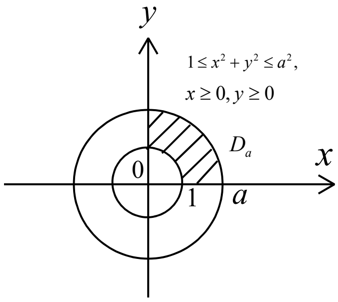


Fig. 5-(1)　　Fig.5-(2)

(1)

$I=\iint_D\frac{x}{x^2+y^2}dxdy$　　$D: 0\le x^2+y^2\le 4, 0\le y\le x$　$x=r\cos\theta, y=r\sin\theta$ とおく.

$\varepsilon>0$ とする.　$D_\varepsilon:\varepsilon^2\le x^2+y^2\le 2^2, 0\le y\le x\to D'_\varepsilon:\varepsilon\le r\le 2, 0\le\theta\le\frac{\pi}{4}$

$$I_\varepsilon=\iint_{D_\varepsilon}\frac{x}{x^2+y^2}dxdy=\iint_{D'_\varepsilon}\frac{r\cos\theta}{r^2}rdrd\theta=(\int_\varepsilon^2 dr)(\int_0^{\frac{\pi}{4}}\cos\theta d\theta)$$

$$I_\varepsilon=\left[\frac{r^2}{2}\right]_\varepsilon^2(\int_0^{\frac{\sqrt{2}}{2}}udu)=(2-\varepsilon)\left[\sin\theta\right]_0^{\frac{\pi}{4}}=\frac{\sqrt{2}(2-\varepsilon)}{2}\qquad I=\lim_{\varepsilon\to 0}I_\varepsilon=\lim_{\varepsilon\to 0}\frac{\sqrt{2}(2-\varepsilon)}{2}=\sqrt{2}$$

(2)

$I=\iiint_D\frac{z}{(x^2+y^2+z^2)^3}dxdydz$ $D: 1\le x^2+y^2+z^2, x\ge 0, y\ge 0, z\ge 0$

$x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$ とおく.

$D_a: 1\le x^2+y^2+z^2\le a^2, x\ge 0, y\ge 0, z\ge 0\to D'_a: 1\le r\le a, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}$

$$I_a=\iiint_{D_a}\frac{z}{(x^2+y^2+z^2)^3}dxdydz=\iiint_{D'_a}\frac{r\cos\theta\cdot r^2\sin\theta}{r^6}drd\theta d\phi=(\int_1^a\frac{1}{r^3}dr)(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta)(\int_0^{\frac{\pi}{2}}d\phi)$$

$$I_a=\left[-\frac{1}{2r^2}\right]_1^a\left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}}\left[\phi\right]_0^{\frac{\pi}{2}}=\frac{1}{8}(\frac{1}{a^2}-1)\cdot(\cos\pi-\cos 0)\frac{\pi}{2}=\frac{\pi}{8}(1-\frac{1}{a^2})$$

$I=\lim_{a\to\infty} I_a=\lim_{a\to\infty}\frac{\pi}{8}(1-\frac{1}{a^2})=\frac{\pi}{8}$

6(10 点×2)

(1)

$I=\iint_D \frac{xy\log(x^2+y^2)}{\sqrt{x^2+y^2}}dxdy \quad D:0\le x^2+y^2\le 4, x\ge 0, y\ge 0 \ \ x=r\cos\theta, y=r\sin\theta$ とおく．$\varepsilon>0$ とする．$D_\varepsilon:\varepsilon^2\le x^2+y^2\le 2^2, x\ge 0, y\ge 0 \to D'_\varepsilon:\varepsilon\le r\le 2, 0\le\theta\le\frac{\pi}{2}$

$$I_\varepsilon=\iint_{D_\varepsilon}\frac{xy\log(x^2+y^2)}{\sqrt{x^2+y^2}}dxdy=\iint_{D'_\varepsilon}\frac{r^2\cos\theta\sin\theta\log r^2}{r}rdrd\theta=(\int_\varepsilon^2 2r^2\log rdr)(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta)$$

$$=(\int_\varepsilon^2 r^2\log rdr)(\int_0^{\frac{\pi}{2}}\sin 2\theta d\theta),\qquad I_1=\int r^2\log rdr=\frac{r^3}{3}\log r-\int\frac{r^3}{3}\cdot\frac{1}{r}dr=\frac{r^3}{3}\log r-\frac{r^3}{9}+c_1$$

$$I_\varepsilon=\left[\frac{r^3}{3}\log r-\frac{r^3}{9}\right]_\varepsilon^2\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}=(\frac{8}{3}\log 2-\frac{8}{9}-\frac{\varepsilon^3}{3}\log\varepsilon+\frac{\varepsilon^3}{9})(-\frac{\cos\pi}{2}+\frac{\cos 0}{2})$$

$$I=\lim_{\varepsilon\to 0}I_\varepsilon=\frac{8}{3}\log 2-\frac{8}{9}-\lim_{\varepsilon\to 0}\frac{\varepsilon^3}{3}\log\varepsilon=\frac{8}{3}\log 2-\frac{8}{9}-\frac{1}{3}\lim_{\varepsilon\to 0}\frac{\log\varepsilon}{\varepsilon^{-3}}=\frac{8}{3}\log 2-\frac{8}{9}-\frac{1}{3}\lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{-3\varepsilon^{-4}}$$

$$=\frac{8}{3}\log 2-\frac{8}{9}+\frac{1}{9}\lim_{\varepsilon\to 0}\varepsilon^3=\frac{8}{3}\log 2-\frac{8}{9}$$

(2)

$I=\iiint_D\frac{ze^{-(x^2+y^2+z^2)}}{x^2+y^2+z^2}dxdydz \quad D:1\le x^2+y^2+z^2, x\ge 0, y\ge 0, z\ge 0$

$x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$ とおく．

$D_a:1\le x^2+y^2+z^2\le a^2, x\ge 0, y\ge 0, z\ge 0\to D'_a:1\le r\le a, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}$

$$I_a=\iiint_{D_a}\frac{ze^{-(x^2+y^2+z^2)}}{x^2+y^2+z^2}dxdydz=\iiint_{D'_a}\frac{r\cos\theta e^{-r^2}}{r^2}r^2\sin\theta drd\theta d\phi=(\int_1^a re^{-r^2}dr)(\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta)(\int_0^{\frac{\pi}{2}}d\phi)$$

$r^2=u$ とおく．$2rdr=du$ である．

$$I_a=(\int_1^{a^2}e^{-u}\frac{du}{2})\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{2}d\theta[\phi]_0^{\frac{\pi}{2}}=\frac{1}{2}\left[-e^{-u}\right]_1^{a^2}\cdot\left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2}=\frac{\pi}{4}(e^{-1}-e^{-a^2})\cdot\frac{1}{2}=\frac{\pi}{8}(e^{-1}-e^{-a^2})$$

$$I=\lim_{a\to\infty}I_a=\lim_{a\to\infty}\frac{\pi}{8}(e^{-1}-e^{-a^2})=\frac{\pi}{8e}$$