数学 I　レポート No.9　解答　2017 年度

1.(2 点×7=14 点)

(1)

$$M=\iint_D \rho(x,y)dxdy, \bar{x}=\frac{1}{M}\iint_D x\rho(x,y)dxdy, \bar{y}=\frac{1}{M}\iint_D y\rho(x,y)dxdy$$ より

① $=\rho(x,y)dxdy$,② $=x\rho(x,y)dxdy$,③ $=y\rho(x,y)dxdy$

(2)

$$M=\iiint_E \rho(x,y,z)dxdydz, \quad \bar{x}=\frac{1}{M}\iiint_E x\rho(x,y,z)dxdydz,$$

$$\bar{y}=\frac{1}{M}\iiint_E y\rho(x,y,z)dxdydz, \quad \bar{z}=\frac{1}{M}\iiint_E z\rho(x,y,z)dxdydz$$ より

④ $=\rho(x,y,z)dxdydz$,⑤ $=x\rho(x,y,z)dxdydz$,⑥ $=y\rho(x,y,z)dxdydz$

⑦ $=z\rho(x,y,z)dxdydz$

2.(6 点+8 点=14 点)

(1)

$\rho=k$ とする.

$$M=\iint_D \rho(x,y)dxdy=k\iint_D dxdy=k\int_1^2 dx\int_{x^2}^{5x^2} dy=k\int_1^2 4x^2dx=k\left[\frac{4}{3}x^3\right]_1^2=\frac{28}{3}k$$

$$\bar{x}=\frac{1}{M}\iint_D x\rho(x,y)dxdy=\frac{1}{M}k\iint_D xdxdy=\frac{1}{M}k\int_1^2 dx\int_{x^2}^{5x^2} xdy$$

$$=\frac{1}{M}k\int_1^2 4x^3dx=\frac{k}{M}\left[x^4\right]_1^2=\frac{15k}{M}=\frac{45}{28}$$

$$\bar{y}=\frac{1}{M}\iint_D y\rho(x,y)dxdy=\frac{1}{M}k\iint_D ydxdy=\frac{1}{M}k\int_1^2 dx\int_{x^2}^{5x^2} ydy$$

$$=\frac{1}{M}k\int_1^2\left[\frac{y^2}{2}\right]_{x^2}^{5x^2}dx=\frac{k}{M}\int_1^2 12x^4dx=\frac{k}{M}\left[\frac{12}{5}x^5\right]_1^2=\frac{12\times 31k}{5M}=\frac{12\times 31\times 3}{5\times 28}=\frac{279}{35}$$

$$G(\bar{x},\bar{y})=(\frac{45}{28},\frac{279}{35})$$

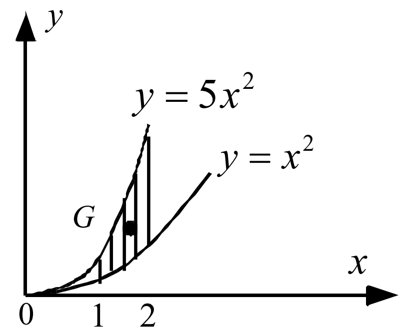

Fig.2-(1)

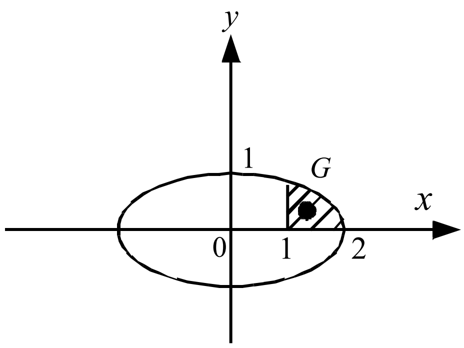

Fig.2-(2)

(2)

$\rho = k$ とする. $\dfrac{x^2}{4}+y^2 \le 1, x \ge 1, y \ge 0$ より

$$M=\iint_D \rho(x,y)dxdy = k\iint_D dxdy = k\int_1^2 dx\int_0^{\sqrt{1-\frac{x^2}{4}}} dy = k\int_1^2 \sqrt{1-\frac{x^2}{4}}dx$$

ここで $x = 2\sin\theta$ とおく. $dx = 2\cos\theta d\theta$　　　$x:1\to 2$ は $\theta:\dfrac{\pi}{6}\to\dfrac{\pi}{2}$ になる.

$$M = 2k\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta d\theta = 2k\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta = k\left[\theta+\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = k(\frac{\pi}{3}-\frac{\sqrt{3}}{4})$$

$$\bar{x} = \frac{1}{M}\iint_D x\rho(x,y)dxdy = \frac{1}{M}k\iint_D xdxdy = \frac{1}{M}k\int_1^2 dx\int_0^{\sqrt{1-\frac{x^2}{4}}} xdy = \frac{k}{M}\int_1^2 x\sqrt{1-\frac{x^2}{4}}dx$$ ここで

$1-\dfrac{x^2}{4} = u$ とおく. $xdx = -2du$　　　$x:1\to 2$ は $u:\dfrac{3}{4}\to 0$ になる.

$$\bar{x} = \frac{k}{M}\int_{\frac{3}{4}}^{0}\sqrt{u}(-2du) = \frac{2k}{M}\int_0^{\frac{3}{4}}\sqrt{u}du = \frac{2k}{M}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{3}{4}} = \frac{\sqrt{3}k}{2M} = \frac{6\sqrt{3}}{4\pi-3\sqrt{3}}$$

$$\bar{y} = \frac{1}{M}\iint_D y\rho(x,y)dxdy = \frac{1}{M}k\iint_D ydxdy = \frac{k}{M}\int_1^2 dx\int_0^{\sqrt{1-\frac{x^2}{4}}} ydy$$

$$= \frac{k}{M}\int_1^2\left[\frac{y^2}{2}\right]_0^{\sqrt{1-\frac{x^2}{4}}}dx = \frac{k}{M}\int_1^2\frac{(1-\frac{x^2}{4})}{2}dx = \frac{k}{2M}\left[x-\frac{x^3}{12}\right]_1^2 = \frac{5k}{24M} = \frac{5}{2(4\pi-3\sqrt{3})}$$

$$G(\bar{x},\bar{y})=(\frac{6\sqrt{3}}{4\pi-3\sqrt{3}},\frac{5}{2(4\pi-3\sqrt{3})})$$

3.(9 点×2=18 点)

(1)

$\rho=k$ とする． $E: x^2+y^2+z^2=1, x\ge 0, y\ge 0, z\ge 0$ は球面座標変換により，

$E': 0\le r\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}$ になる．

$$M=\iiint_E \rho dxdydz = k\iiint_{E'} r^2\sin\theta drd\theta d\phi = k(\int_0^1 r^2 dr)(\int_0^{\frac{\pi}{2}}\sin\theta d\theta)(\int_0^{\frac{\pi}{2}} d\phi)$$

$$M=k\left[\frac{r^3}{3}\right]_0^1\left[-\cos\theta\right]_0^{\frac{\pi}{2}}\left[\phi\right]_0^{\frac{\pi}{2}}=k\cdot\frac{1}{3}\cdot 1\cdot\frac{\pi}{2}=\frac{1}{6}k\pi$$

$$\bar{x}=\frac{1}{M}\iiint_E \rho x dxdydz=\frac{k}{M}\iiint_{E'} r\sin\theta\cos\phi r^2\sin\theta drd\theta d\phi=\frac{k}{M}(\int_0^1 r^3 dr)(\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta)(\int_0^{\frac{\pi}{2}}\cos\phi d\phi)$$

$$=\frac{k}{M}\left[\frac{r^4}{4}\right]_0^1\cdot\frac{\pi}{4}\cdot 1=\frac{k\pi}{16M}=\frac{6k\pi}{16k\pi}=\frac{3}{8}$$

幾何学的対称性から $\bar{x}=\bar{y}=\bar{z}$ である．

$$G(\bar{x},\bar{y},\bar{z})=(\frac{3}{8},\frac{3}{8},\frac{3}{8})$$

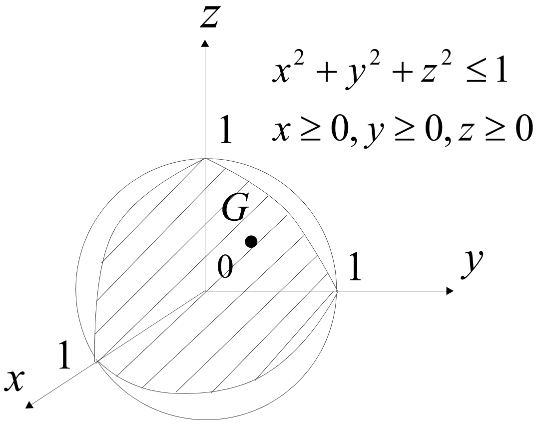

Fig.3-(1)

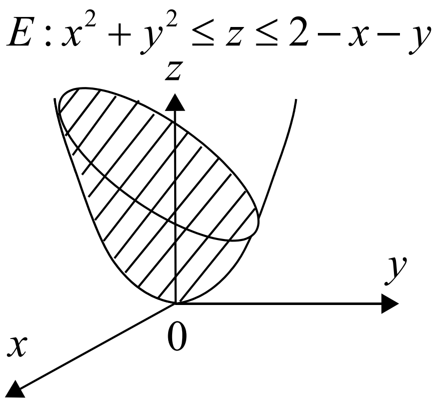

Fig.3-(2)

(2)

$\rho=k$ とする． $E: x^2+y^2\le z\le 2-x-y$　$z=x^2+y^2, z=2-x-y$ から xy 平面に対する

正射影は $x^2+y^2=2-x-y$ である． $x^2+y^2+x+y\le 2$ から正射影は

$D:(x+\frac{1}{2})^2+(y+\frac{1}{2})^2 \le (\sqrt{\frac{5}{2}})^2$ である．

$$M=\iiint_E \rho dxdydz = k\iint_D dxdy\int_{x^2+y^2}^{2-x-y} dz = k\iint_D dxdy(2-x-y-x^2-y^2)$$

$=k\iint_D dxdy\{\frac{5}{2}-(x+\frac{1}{2})^2-(y+\frac{1}{2})^2\}$ ここで $x+\frac{1}{2}=u, y+\frac{1}{2}=v$ とおき極座標変換を使う．

$$=k\iint_{D'} dudv(\frac{5}{2}-u^2-v^2)=k\int_0^{\sqrt{\frac{5}{2}}} dr\int_0^{2\pi} d\theta(\frac{5}{2}-r^2)r = 2\pi k\int_0^{\sqrt{\frac{5}{2}}}(\frac{5}{2}r-r^3)dr = 2\pi k\left[\frac{5}{4}r^2-\frac{1}{4}r^4\right]_0^{\sqrt{\frac{5}{2}}}$$

$$=2\pi k(\frac{5}{4}\cdot\frac{5}{2}-\frac{1}{4}\cdot\frac{25}{4})=\frac{25}{8}\pi k$$

$$\bar{x}=\frac{1}{M}\iiint_E x\rho dxdydz=\frac{k}{M}\iint_D dxdyx\int_{x^2+y^2}^{2-x-y} dz = \frac{k}{M}\iint_D dxdyx(2-x-y-x^2-y^2)$$

$$=\frac{k}{M}\iint_D dxdyx\{\frac{5}{2}-(x+\frac{1}{2})^2-(y+\frac{1}{2})^2\}=\frac{k}{M}\iint_{D'}(u-\frac{1}{2})(\frac{5}{2}-u^2-v^2)dudv$$

$$=\frac{k}{M}\int_0^{2\pi} d\theta\int_0^{\sqrt{\frac{5}{2}}}(r\cos\theta-\frac{1}{2})(\frac{5}{2}-r^2)rdr=\frac{k}{M}\int_0^{\sqrt{\frac{5}{2}}}\{r^2(\frac{5}{2}-r^2)\int_0^{2\pi}\cos\theta d\theta-\frac{1}{2}(\frac{5}{2}-r^2)r\int_0^{2\pi}d\theta\}dr$$

$$=\frac{k}{M}\int_0^{\sqrt{\frac{5}{2}}}-\frac{2\pi}{2}(\frac{5}{2}-r^2)rdr=-\frac{k\pi}{M}\int_0^{\sqrt{\frac{5}{2}}}(\frac{5}{2}r-r^3)dr=-\frac{k\pi}{M}\left[\frac{5r^2}{4}-\frac{1}{4}r^4\right]_0^{\sqrt{\frac{5}{2}}}=-\frac{k\pi}{M}(\frac{5}{4}\cdot\frac{5}{2}-\frac{1}{4}\cdot\frac{25}{4})$$

$$=-\frac{k\pi}{M}\cdot\frac{25}{16}=-\frac{25\pi k\cdot 8}{25\pi k\cdot 16}=-\frac{1}{2}$$

$$\bar{y}=\frac{1}{M}\iiint_E y\rho dxdydz=\frac{k}{M}\iint_D dxdyy\int_{x^2+y^2}^{2-x-y} dz = \frac{k}{M}\iint_D dxdyy(2-x-y-x^2-y^2)$$

$$=\frac{k}{M}\iint_D dxdyy\{\frac{5}{2}-(x+\frac{1}{2})^2-(y+\frac{1}{2})^2\}=\frac{k}{M}\iint_{D'}(v-\frac{1}{2})(\frac{5}{2}-u^2-v^2)dudv$$

$$=\frac{k}{M}\int_0^{2\pi} d\theta\int_0^{\sqrt{\frac{5}{2}}}(r\sin\theta-\frac{1}{2})(\frac{5}{2}-r^2)rdr=\frac{k}{M}\int_0^{\sqrt{\frac{5}{2}}}\{r^2(\frac{5}{2}-r^2)\int_0^{2\pi}\sin\theta d\theta-\frac{1}{2}(\frac{5}{2}-r^2)r\int_0^{2\pi}d\theta\}dr$$

$$=\frac{k}{M}\int_0^{\sqrt{\frac{5}{2}}}-\frac{2\pi}{2}(\frac{5}{2}-r^2)rdr=-\frac{k\pi}{M}\int_0^{\sqrt{\frac{5}{2}}}(\frac{5}{2}r-r^3)dr=-\frac{k\pi}{M}\left[\frac{5r^2}{4}-\frac{1}{4}r^4\right]_0^{\sqrt{\frac{5}{2}}}=-\frac{k\pi}{M}(\frac{5}{4}\cdot\frac{5}{2}-\frac{1}{4}\cdot\frac{25}{4})$$

$$=-\frac{k\pi}{M}\cdot\frac{25}{16}=-\frac{25\pi k\cdot 8}{25\pi k\cdot 16}=-\frac{1}{2}$$

$$\bar{z}=\frac{1}{M}\iiint_E z\rho dxdydz=\frac{k}{M}\iint_D dxdyy\int_{x^2+y^2}^{2-x-y}zdz=\frac{k}{M}\iint_D dxdy\left[\frac{z^2}{2}\right]_{x^2+y^2}^{2-x-y}$$

$$=\frac{k}{M}\iint_D\frac{(2-x-y)^2-(x^2+y^2)^2}{2}dxdy$$

ここで$u=x+\frac{1}{2},v=y+\frac{1}{2}$とおく．$x=u-\frac{1}{2},y=v-\frac{1}{2}$を代入する．

$$\bar{z}=\frac{k}{2M}\iint_{D'}\{(3-u-v)^2-(u^2+v^2-u-v+\frac{1}{2})^2\}dudv$$ $u=r\cos\theta,v=r\sin\theta$に変換する．

$$\bar{z}=\frac{k}{2M}\iint_{D''}\{(3-r\cos\theta-r\sin\theta)^2-(r^2+\frac{1}{2}-r\cos\theta-r\sin\theta)^2\}rdrd\theta$$

$$=\frac{k}{2M}\iint_{D''}\{(r\cos\theta+r\sin\theta)^2-6(r\cos\theta+r\sin\theta)+9$$

$$-(r\cos\theta+r\sin\theta)^2+2(r^2+\frac{1}{2})(r\cos\theta+r\sin\theta)-(r^2+\frac{1}{2})^2\}rdrd\theta$$

$$\bar{z}=\frac{k}{2M}\iint_{D'''}\{(2r^2-5)r(\cos\theta+\sin\theta)+9-(r^2+\frac{1}{2})^2\}rdrd\theta$$

$$=\frac{k}{2M}\int_0^{2\pi}d\theta\int_0^{\sqrt{\frac{5}{2}}}\{(2r^2-5)r(\cos\theta+\sin\theta)+9-(r^2+\frac{1}{2})^2\}rdr$$

ここで

$\int_0^{2\pi}\sin\theta d\theta=\int_0^{2\pi}\cos\theta d\theta=0$を使う．

$$\bar{z}=\frac{k}{2M}\int_0^{2\pi}d\theta\int_0^{\sqrt{\frac{5}{2}}}\{9-(r^2+\frac{1}{2})^2\}rdr=\frac{2\pi k}{2M}\int_0^{\sqrt{\frac{5}{2}}}\{9r-(r^4+r^2+\frac{1}{4})r\}dr$$

$$=\frac{\pi k}{M}\int_0^{\sqrt{\frac{5}{2}}}(\frac{35}{4}r-r^3-r^5)dr=\frac{\pi k}{M}\left[\frac{35}{8}r^2-\frac{r^4}{4}-\frac{r^6}{6}\right]_0^{\sqrt{\frac{5}{2}}}=\frac{\pi k}{M}(\frac{35}{8}\cdot\frac{5}{2}-\frac{1}{4}\cdot\frac{25}{4}-\frac{1}{6}\cdot\frac{125}{8})$$

$$=\frac{325}{48}\cdot\frac{\pi k}{M}=\frac{325\pi k}{48}\cdot\frac{8}{25\pi k}=\frac{13}{6}$$

$$G(\bar{x},\bar{y},\bar{z})=(-\frac{1}{2},-\frac{1}{2},\frac{13}{6})$$

4.(9 点×2=18 点)

(1)

$E:x^2+y^2+z^2\le 1,y\ge 0,z\ge 0$に球面座標変換を使えば

$E' : 0 \le r \le 1, 0 \le \theta \le \dfrac{\pi}{2}, 0 \le \phi \le \pi$ となる.

$$M = \iiint_E \rho dxdydz = \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi} d\phi r^2 \sin\theta \cdot \frac{k}{r} = k(\int_0^1 rdr)(\int_0^{\frac{\pi}{2}} \sin\theta d\theta)(\int_0^{\pi} d\phi) = k \cdot \frac{1}{2} \cdot 1 \cdot \pi = \frac{\pi k}{2}$$

$$\bar{x} = \frac{1}{M}\iiint_E x\rho dxdydz = \frac{1}{M}\iiint_{E'} \frac{k}{r} \cdot r^2 \sin\theta \cdot r\sin\theta\cos\phi drd\theta d\phi$$

$$= \frac{k}{M}(\int_0^1 r^2 dr)(\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta)(\int_0^{\pi} \cos\phi d\phi) = \frac{k}{M}\cdot\frac{1}{3}\cdot\frac{\pi}{4}\cdot[\sin\phi]_0^{\pi} = 0$$

$$\bar{y} = \frac{1}{M}\iiint_E y\rho dxdydz = \frac{1}{M}\iiint_{E'} \frac{k}{r} \cdot r^2 \sin\theta \cdot r\sin\theta\sin\phi drd\theta d\phi$$

$$= \frac{k}{M}(\int_0^1 r^2 dr)(\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta)(\int_0^{\pi} \sin\phi d\phi) = \frac{k}{M}\cdot\frac{1}{3}\cdot\frac{\pi}{4}\cdot[-\cos\phi]_0^{\pi} = \frac{k\pi}{6M} = \frac{\pi k}{3\pi k} = \frac{1}{3}$$

$$\bar{z} = \frac{1}{M}\iiint_E z\rho dxdydz = \frac{1}{M}\iiint_{E'} \frac{k}{r} \cdot r^2 \sin\theta \cdot r\cos\theta drd\theta d\phi$$

$$= \frac{k}{M}(\int_0^1 r^2 dr)(\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta)(\int_0^{\pi} d\phi) = \frac{k}{M}\cdot\frac{1}{3}(\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} d\theta)\pi = \frac{k\pi}{3M}\left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi k}{6M} = \frac{\pi k}{3\pi k} = \frac{1}{3}$$

$$G(\bar{x}, \bar{y}, \bar{z}) = (0, \frac{1}{3}, \frac{1}{3})$$

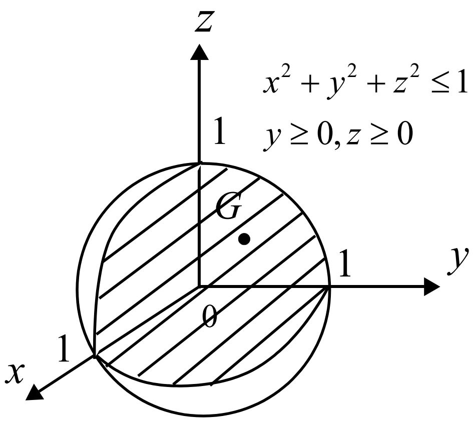

Fig.4-(1)

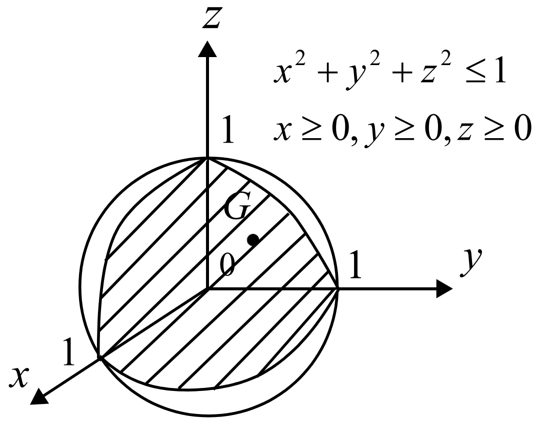

Fig.4-(2)

(2)

$E : x^2 + y^2 + z^2 \le 1, x \ge 0, y \ge 0, z \ge 0$ に球面座標変換を使えば

$E' : 0 \le r \le 1, 0 \le \theta \le \dfrac{\pi}{2}, 0 \le \phi \le \dfrac{\pi}{2}$ となる.

$$M=\iiint_E \rho dxdydz=\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} d\phi r^2\sin\theta\cdot\frac{k}{r}=k(\int_0^1 rdr)(\int_0^{\frac{\pi}{2}}\sin\theta d\theta)(\int_0^{\frac{\pi}{2}} d\phi)=k\cdot\frac{1}{2}\cdot 1\cdot\frac{\pi}{2}=\frac{\pi k}{4}$$

$$\bar{x}=\frac{1}{M}\iiint_E x\rho dxdydz=\frac{1}{M}\iiint_{E'}\frac{k}{r}\cdot r^2\sin\theta\cdot r\sin\theta\cos\phi drd\theta d\phi$$

$$=\frac{k}{M}(\int_0^1 r^2dr)(\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta)(\int_0^{\frac{\pi}{2}}\cos\phi d\phi)=\frac{k}{M}\cdot\frac{1}{3}\cdot\frac{\pi}{4}\cdot[\sin\phi]_0^{\frac{\pi}{2}}=\frac{\pi k}{12M}=\frac{\pi k}{3\pi k}=\frac{1}{3}$$

$$\bar{y}=\frac{1}{M}\iiint_E y\rho dxdydz=\frac{1}{M}\iiint_{E'}\frac{k}{r}\cdot r^2\sin\theta\cdot r\sin\theta\sin\phi drd\theta d\phi$$

$$=\frac{k}{M}(\int_0^1 r^2dr)(\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta)(\int_0^{\frac{\pi}{2}}\sin\phi d\phi)=\frac{k}{M}\cdot\frac{1}{3}\cdot\frac{\pi}{4}\cdot[-\cos\phi]_0^{\frac{\pi}{2}}=\frac{\pi k}{12M}=\frac{\pi k}{3\pi k}=\frac{1}{3}$$

$$\bar{z}=\frac{1}{M}\iiint_E z\rho dxdydz=\frac{1}{M}\iiint_{E'}\frac{k}{r}\cdot r^2\sin\theta\cdot r\cos\theta drd\theta d\phi$$

$$=\frac{k}{M}(\int_0^1 r^2dr)(\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta)(\int_0^{\frac{\pi}{2}} d\phi)=\frac{k}{M}\cdot\frac{1}{3}(\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}d\theta)\cdot\frac{\pi}{2}=\frac{\pi k}{6M}\left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi k}{12M}=\frac{\pi k}{3\pi k}=\frac{1}{3}$$

$$G(\bar{x},\bar{y},\bar{z})=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$$

5.(2 点×9＝18 点)

(1)

$$I_l=\iint_D \rho(x,y)p^2(x,y)dxdy,\ \ I_x=\iint_D \rho(x,y)y^2dxdy,\ \ I_y=\iint_D \rho(x,y)x^2dxdy,$$

$I_z=\iint_D \rho(x,y)(x^2+y^2)dxdy$ より

①$=\rho(x,y)p^2(x,y)dxdy$　（3 点）　②$=\rho(x,y)y^2dxdy$　③$=\rho(x,y)x^2dxdy$

④$=\rho(x,y)(x^2+y^2)dxdy$

(2)

$$I_l=\iiint_E \rho(x,y,z)p^2(x.y.z)dxdydz,\ \ I_x=\iiint_E \rho(x,y,z)(y^2+z^2)dxdydz,$$

$I_y=\iiint_E \rho(x,y,z)(x^2+z^2)dxdydz,\ \ I_z=\iiint_E \rho(x,y,z)(x^2+y^2)dxdydz$　より

⑤$=\rho(x,y,z)p^2(x,y,z)dxdydz$　（3 点）　⑥$=\rho(x,y,z)(y^2+z^2)dxdydz$

⑦$=\rho(x,y,z)(x^2+z^2)dxdydz$　⑧$=\rho(x,y,z)(x^2+y^2)dxdydz$

(3)

$I_l = I_0 + a^2 M$ より ⑨$= a^2 M$

6.(9 点×2＝18 点)

(1)

$$I_x = \iint_D \rho(x,y) y^2 dxdy = k\int_0^a dx \int_0^{x\sqrt{x}} y^2 dy = k\int_0^a dx \left[\frac{y^3}{3}\right]_0^{x\sqrt{x}} = \frac{k}{3}\int_0^a x^{\frac{9}{2}} dx = \frac{k}{3}\left[\frac{x^{\frac{11}{2}}}{\frac{11}{2}}\right]_0^a = \frac{2k}{33} a^5 \sqrt{a}$$

$$I_y = \iint_D \rho(x,y) x^2 dxdy = k\int_0^a dx \int_0^{x\sqrt{x}} x^2 dy = k\int_0^a x^3 \sqrt{x} dx = k\int_0^a x^{\frac{7}{2}} dx = k\left[\frac{x^{\frac{9}{2}}}{\frac{9}{2}}\right]_0^a = \frac{2k}{9} a^4 \sqrt{a}$$

$$I_z = \iint_D \rho(x,y)(x^2+y^2) dxdy = I_x + I_y = \frac{2k}{33} a^5 \sqrt{a} + \frac{2k}{9} a^4 \sqrt{a}$$

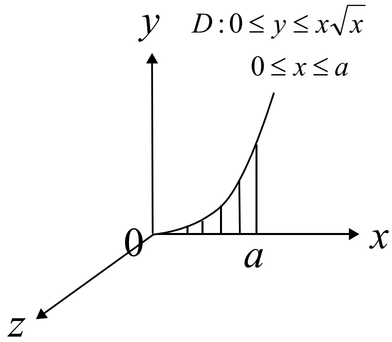

Fig.6-(1)

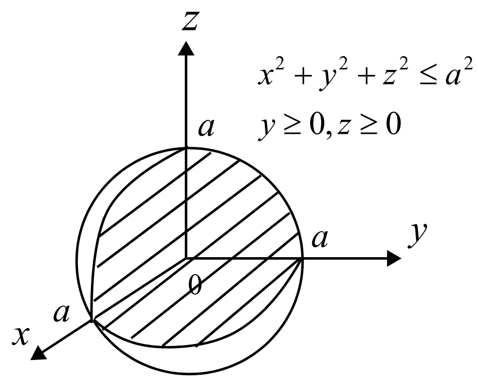

Fig.6-(2)

(2)

$E : x^2 + y^2 + z^2 \le a^2, y \ge 0, z \ge 0$ に球面座標変換を使えば,

$E' : 0 \le r \le a, 0 \le \theta \le \dfrac{\pi}{2}, 0 \le \phi \le \pi$ になる. $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$$I_x = \iiint_E \rho(x,y,z)(y^2+z^2) dxdydz = k\int_0^a dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi} d\phi (y^2+z^2) r^2 \sin\theta$$

$$= k\int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\phi\{(r\sin\theta\sin\phi)^2 + (r\cos\theta)^2\}r^2\sin\theta$$

$$= k(\int_0^a r^4 dr)\{\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta\int_0^{\pi} \sin^2\phi d\phi + \pi\int_0^{\frac{\pi}{2}} (\sin\theta - \sin^3\theta)d\theta\} = k\frac{a^5}{5}\{\frac{2}{3}\cdot 2\int_0^{\frac{\pi}{2}} \sin^2\phi d\phi + 2\pi(1-\frac{2}{3})\}$$

$$= k\frac{a^5}{5}\{\frac{2}{3}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 2\pi(1-\frac{2}{3})\} = \frac{ka^5}{5}(\frac{\pi}{3}+\frac{\pi}{3}) = \frac{2\pi ka^5}{15}$$

$$I_y = \iiint_E \rho(x,y,z)(x^2+z^2)dxdydz = k\int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\phi(x^2+z^2)r^2\sin\theta$$

$$= k\int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\phi\{(r\sin\theta\cos\phi)^2 + (r\cos\theta)^2\}r^2\sin\theta$$

$$= k(\int_0^a r^4 dr)\{\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta\int_0^{\pi} \cos^2\phi d\phi + \pi\int_0^{\frac{\pi}{2}} (\sin\theta - \sin^3\theta)d\theta\} = k\frac{a^5}{5}\{\frac{2}{3}\cdot 2\int_0^{\frac{\pi}{2}} \cos^2\phi d\phi + 2\pi(1-\frac{2}{3})\}$$

$$= k\frac{a^5}{5}\{\frac{2}{3}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 2\pi(1-\frac{2}{3})\} = \frac{ka^5}{5}(\frac{\pi}{3}+\frac{\pi}{3}) = \frac{2\pi ka^5}{15}$$

$$I_z = \iiint_E \rho(x,y,z)(x^2+y^2)dxdydz = k\int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\phi(x^2+y^2)r^2\sin\theta$$

$$= k\int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\phi\{(r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2\}r^2\sin\theta$$

$$= k(\int_0^a r^4 dr)\{\int_0^{\frac{\pi}{2}} \sin^3\theta d\theta\int_0^{\pi} \cos^2\phi d\phi + \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta\int_0^{\pi} \sin^2\phi d\phi\} = k\frac{a^5}{5}\cdot\frac{2}{3}(2\int_0^{\frac{\pi}{2}} \cos^2\phi d\phi + 2\int_0^{\frac{\pi}{2}} \sin^2\phi d\phi)$$

$$= \frac{2ka^5}{15}\cdot 2\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{2\pi ka^5}{15}$$

$$I_x = I_y = I_z = \frac{2\pi ka^5}{15}$$