2017.2.2

電気電子工学科

数学Ⅲ　　期末試験　解答

.

1.(5点×2＝10点)

(1)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},u(x,y)=x^3-3xy^2+2x,f(0)=0$$

$$\frac{\partial u}{\partial x}=3x^2-3y^2+2=\frac{\partial v}{\partial y},v=\int(3x^2-3y^2+2)dy=3x^2y-y^3+2y+C_1(x),$$

$$\frac{\partial u}{\partial y}=-6xy=-\frac{\partial v}{\partial x}=-\{6xy+C_1'(x)\},C_1'(x)=0,C_1(x)=C$$

$v=3x^2y-y^3+2y+C,v(0,0)=C=0$より，$v(x,y)=3x^2y-y^3+2y$

$$f(x,y)=x^3-3xy^2+2x+i(3x^2y-y^3+2y),y\to 0,x\to z,f(z)=z^3+2z$$

(2)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},u(x,y)=e^x(x\cos y+\cos y-y\sin y),f(0)=1$$

$$\frac{\partial u}{\partial x}=e^x(x\cos y+\cos y-y\sin y)+e^x\cos y=e^x(x\cos y+2\cos y-y\sin y)=\frac{\partial v}{\partial y}$$

$$v=\int e^x(x\cos y+2\cos y-y\sin y)dy=e^x(x\sin y+2\sin y)-e^x\int y\sin ydy+C_1(x)$$

$$u_1=y,v_1'=\sin y,u_1'=1,v_1=-\cos y,\int y\sin ydy=-y\cos y+\int\cos ydy=-y\cos y+\sin y$$

$$v=e^x(x\sin y+2\sin y)-e^x(-y\cos y+\sin y)+C_1(x)=e^x(x\sin y+y\cos y+\sin y)+C_1(x)$$

$$\frac{\partial u}{\partial y}=e^x(-x\sin y-\sin y-\sin y-y\cos y)=e^x(-x\sin y-2\sin y-y\cos y)$$

$$=-\frac{\partial v}{\partial x}=-e^x(x\sin y+y\cos y+\sin y)-e^x\sin y-C_1'(x)=-e^x(x\sin y+y\cos y+2\sin y)-C_1'(x),$$

$$C_1'(x)=0,C_1(x)=C,v(x,y)=e^x(x\sin y+y\cos y+\sin y)+C,f(0)=u(0,0)+iv(0,0)=1+iC=1,C=0$$

$$v(x,y)=e^x(x\sin y+y\cos y+\sin y),f(z)=e^x(x\cos y+\cos y-y\sin y)+ie^x(x\sin y+y\cos y+\sin y),$$

$y\to 0,x\to z$と置き換える　$f(z)=(z+1)e^z$

2.(5点$\times$2＝10点）

(1)

$$J=\oint_C \frac{1}{2z^2+4z+5}dz, C:|z|=2, 2z^2+4z+5=0, z=\frac{-2\pm\sqrt{4-10}}{2}=\frac{-2\pm\sqrt{6}i}{2}$$

$$c_1=\frac{-2+\sqrt{6}i}{2}\in D, c_2=\frac{-2-\sqrt{6}i}{2}\in D, N=2$$

$$J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j}(\frac{1}{2z^2+4z+5})dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)}{2z^2+4z+5}\}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}(\frac{1}{4z+4})$$

$$=\frac{\pi i}{2}(\frac{1}{c_1+1}+\frac{1}{c_2+1})=\frac{\pi i}{2}(\frac{1}{-1+\frac{\sqrt{6}}{2}i+1}+\frac{1}{-1-\frac{\sqrt{6}}{2}i+1})=0$$

(2)

$$J=\oint_C \frac{1}{z^4+3z^2+2}dz, C:|z-3i|=3, z^4+3z^2+2=(z^2+1)(z^2+2)=0, z=\pm i, z=\pm\sqrt{2}i$$

$$z=i=c_1\in D, z=\sqrt{2}i=c_2\in D, N=2$$

$$J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j}(\frac{1}{z^4+3z^2+2})dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)}{z^4+3z^2+2}\}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}(\frac{1}{4z^3+6z})$$

$$=\pi i(\frac{1}{2c_1^3+3c_1}+\frac{1}{2c_2^3+3c_2})=\pi i(\frac{1}{2i^3+6i}+\frac{1}{2(\sqrt{2})^3i^3+3\sqrt{2}i})=\pi(\frac{1}{-2+3}+\frac{1}{-4\sqrt{2}+3\sqrt{2}})$$

$$=\pi(1-\frac{1}{\sqrt{2}})=\frac{2-\sqrt{2}}{2}\pi$$

3.(10点$\times$2＝20点）

(1)

$$J=\int_{-\infty}^{\infty}\frac{1}{x^2+2x+6}dx, f(z)=\frac{1}{z^2+2z+6}, Rf(\mathrm{Re}^{i\theta})\to 0(R\to\infty), (0\le\theta\le\pi)$$

$$z^2+2z+6=0, z=-1\pm\sqrt{1-6}=-1\pm\sqrt{5}i, c_1=-1+\sqrt{5}i\in D$$

$$J=2\pi i\operatorname*{Res}_{z=c_1}(\frac{1}{z^2+2z+6})=2\pi i\lim_{z\to c_1}\frac{(z-c_1)}{z^2+2z+6}=2\pi i\lim_{z\to c_1}\frac{1}{2z+2}=\frac{\pi i}{c_1+1}=\frac{\pi}{\sqrt{5}}=\frac{\sqrt{5}}{5}\pi$$

(2)

$$J=\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}dx, f(z)=\frac{x^2}{z^4+1}, Rf(\mathrm{Re}^{i\theta})\to 0(R\to\infty), (0\le\theta\le\pi)$$

$$z^4+1=0, z^4=-1=e^{\pi i+2n\pi i}, z=e^{\frac{\pi i+2n\pi i}{4}}, n=0,1,2,3, z_0=e^{\frac{\pi i}{4}}, z_1=e^{\frac{3\pi i}{4}}, z_2=e^{\frac{5\pi i}{4}}, z_2=e^{\frac{7\pi i}{4}}$$

$$c_1=e^{\frac{\pi i}{4}}\in D, c_2=e^{\frac{3\pi i}{4}}\in D, N=2$$

$$J = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=c_j} s(f(z))dz = 2\pi i \sum_{j=1}^{N} \lim_{z\to c_j} \{\frac{(z-c_j)z^2}{z^4+1}\} = 2\pi i \sum_{j=1}^{N} \lim_{z\to c_j} \{\frac{z^2+(z-c_j)2z}{4z^3}\} = 2\pi i (\frac{1}{4c_1}+\frac{1}{4c_2})$$

$$= \frac{\pi i}{2}(e^{-\frac{\pi i}{4}} + e^{-\frac{3\pi i}{4}}) = \frac{\pi i}{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}\pi$$

4.(10点×2＝20点）

(1)

$$S = \int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2+2x+7} dx, f(z) = \frac{1}{z^2+2z+7}, m = \pi > 0, f(\mathrm{Re}^{i\theta}) \to 0 (R \to \infty), (0 \le \theta \le \pi)$$

$$z^2 + 2z + 7 = 0, z = -1 \pm \sqrt{1-7} = -1 \pm \sqrt{6} i, c_1 = -1 + \sqrt{6} i \in D$$

$$J = S + iT = 2\pi i \operatorname*{Res}_{z=c_1} s(\frac{e^{i\pi z}}{z^2+2z+7}) = 2\pi i \lim_{z\to c_1} \frac{(z-c_1)e^{i\pi z}}{z^2+2z+7} = 2\pi i \lim_{z\to c_1} \frac{e^{i\pi z} + (z-c_1)e^{i\pi z} i\pi}{2z+2} = \frac{\pi i e^{i\pi c_1}}{c_1+1}$$

$$= \frac{\pi e^{i\pi(-1+\sqrt{6}i)}}{\sqrt{6}} = \frac{\sqrt{6}}{6}\pi(-e^{-\sqrt{6}\pi}) = -\frac{\sqrt{6}}{6}\pi e^{-\sqrt{6}\pi}, S = \mathrm{Re}(J) = -\frac{\sqrt{6}}{6}\pi e^{-\sqrt{6}\pi}$$

(2)

$$T = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^4+5x^2+4} dx, f(z) = \frac{z}{z^4+5z^2+4}, m = \pi > 0, f(\mathrm{Re}^{i\theta}) \to 0 (R \to \infty), (0 \le \theta \le \pi)$$

$$z^4 + 5z^2 + 4 = (z^2+1)(z^2+4) = 0, z = \pm i, z = \pm 2i, c_1 = i \in D, c_2 = 2i \in D, N = 2$$

$$J = S + iT = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=c_j} s(\frac{z e^{i\pi z}}{z^4+5z^2+4})dz = 2\pi i \sum_{j=1}^{N} \lim_{z\to c_j} \frac{(z-c_j) z e^{i\pi z}}{z^4+5z^2+4}$$

$$= 2\pi i \sum_{j=1}^{N} \lim_{z\to c_j} \frac{(2z-c_j)e^{i\pi z} + (z-c_j) z e^{i\pi z} \times i\pi}{4z^3+10z}$$

$$= 2\pi i (\frac{c_1 e^{i\pi c_1}}{4c_1^3+10c_1} + \frac{c_2 e^{i\pi c_2}}{4c_2^3+10c_2}) = 2\pi i (\frac{i e^{i\pi i}}{4i^3+10i} + \frac{2i e^{i2\pi i}}{32i^3+20i})$$

$$= -\pi(\frac{e^{-\pi}}{-2i+5i} + \frac{e^{-2\pi}}{-8i+5i}) = -\pi(\frac{e^{-\pi}}{3i} - \frac{e^{-2\pi}}{3i}) = \frac{\pi}{3}(e^{-\pi} - e^{-2\pi})i,$$

$$T = \mathrm{Im}(J) = \frac{\pi}{3}(e^{-\pi} - e^{-2\pi})$$

5.(10点×2＝20点）

(1)

$$J = \int_0^{2\pi} \frac{1}{3+\cos\theta} d\theta, z = e^{i\theta}, dz = e^{i\theta} i d\theta = z i d\theta, J = \oint_{|z|=1} \frac{1}{3+\frac{z+z^{-1}}{2}} \frac{dz}{zi} = \oint_{|z|=1} \frac{2}{i} \frac{1}{z^2+6z+1} dz,$$

$$z^2 + 6z + 1 = 0, z = -3 \pm \sqrt{9-1} = -3 \pm \sqrt{8} = -3 \pm 2\sqrt{2}, c_1 = -3 + 2\sqrt{2},$$

$$J = 2\pi i \operatorname*{Res}_{z=c_1} s(\frac{2}{i}\frac{1}{z^2+6z+1}) = 4\pi \lim_{z\to c_1} \{\frac{(z-c_1)}{z^2+6z+1}\} = 4\pi \lim_{z\to c_1} (\frac{1}{2z+6}) = \frac{4\pi}{-6+4\sqrt{2}+6} = \frac{\sqrt{2}}{2}\pi$$

(2)

$$J=\int_0^{2\pi}\frac{2+\cos\theta}{3+2\sin\theta}d\theta, z=e^{i\theta}, dz=e^{i\theta}id\theta=zid\theta, J=\oint_{|z|=1}\frac{2+\frac{z+z^{-1}}{2}}{3+2\frac{z-z^{-1}}{2i}}\frac{dz}{zi}=\oint_{|z|=1}\frac{4i+iz+iz^{-1}}{6i+2z-2z^{-1}}\times\frac{1}{zi}dz$$

$$=\oint_{|z|=1}\frac{4z+z^2+1}{(2z^2+6iz-2)z}dz=\frac{1}{2}\oint_{|z|=1}\frac{z^2+4z+1}{z(z^2+3zi-1)}dz, z^2+3zi-1=0, z=\frac{-3i\pm\sqrt{-9+4}}{2}=\frac{-3i\pm\sqrt{5}i}{2}$$

$$c_1=0\in D, c_2=\frac{-3+\sqrt{5}}{2}i\in D, N=2$$

$$J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j}\{\frac{z^2+4z+1}{2z(z^2+3zi-1)}\}=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)(z^2+4z+1)}{z^3+3iz^2-z}\}$$

$$=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z^2+4z+1)+(z-c_j)(2z+4)}{3z^2+6iz-1}\}=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}(\frac{z^2+4z+1}{3z^2+6iz-1}), c_2^2=-3c_2i+1$$

$$J=\pi i(-1+\frac{c_2^2+4c_2+1}{3c_2^2+6ic_2-1})=\pi i(-1+\frac{-3ic_2+1+4c_2+1}{-9c_2i+3+6ic_2-1})=\pi i(-1+\frac{-3ic_2+1+4c_2+1}{-3c_2i+2})$$

$$=\pi i(\frac{3c_2i-2-3ic_2+1+4c_2+1}{-3c_2i+2})=\pi i\frac{4c_2}{-3c_2i+2}=\frac{-4\frac{-3+\sqrt{5}}{2}}{2+3\frac{-3+\sqrt{5}}{2}}\pi=\frac{12-4\sqrt{5}}{4-9+3\sqrt{5}}\pi=\frac{4(3-\sqrt{5})}{3\sqrt{5}-5}\pi$$

$$=\frac{4(3-\sqrt{5})}{\sqrt{5}(3-\sqrt{5})}\pi=\frac{4}{\sqrt{5}}\pi=\frac{4}{5}\sqrt{5}\pi$$

6.(10点$\times$2＝20点）

(1)

$$f(z)=\frac{3z+2}{z^2+7z+10}, D=\{z|3<|z|<4\},$$

$$f(z)=\frac{3z+2}{(z+2)(z+5)}=\frac{\alpha_1}{z+2}+\frac{\alpha_2}{z+5}, \alpha_1=\lim_{z\to-2}\frac{(z+2)(3z+2)}{(z+2)(z+5)}=-\frac{4}{3}, \alpha_2=\lim_{z\to-5}\frac{(z+5)(3z+2)}{(z+2)(z+5)}=\frac{13}{3}$$

$$f(z)=\frac{\alpha_1}{z(1+\frac{2}{z})}+\frac{\alpha_2}{5(1+\frac{z}{5})}=\frac{\alpha_1}{z}\sum_{n=0}^{\infty}(-\frac{2}{z})^n+\frac{\alpha_2}{5}\sum_{n=0}^{\infty}(-\frac{z}{5})^n=\sum_{n=0}^{\infty}-\frac{4}{3}(-2)^n z^{-(n+1)}+\sum_{n=0}^{\infty}\frac{13}{15}(-\frac{1}{5})^n z^n, n+1=n'$$

$$f(z)=\sum_{n'=1}^{\infty}\frac{2}{3}(-2)^{n'}z^{-n'}+\sum_{n=0}^{\infty}\frac{13}{15}(-\frac{1}{5})^n z^n=\sum_{n=1}^{\infty}\frac{2}{3}(-2)^n z^{-n}+\sum_{n=0}^{\infty}\frac{13}{15}(-\frac{1}{5})^n z^n$$

(2)

$$f(z)=\frac{3z^2+z+2}{(z^2+1)(z+4)}, D=\{z \big| 2<|z|<3\},$$

$$f(z)=\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{\alpha_1 z+\beta_1}{z^2+1}+\frac{\alpha_2}{z+4}=\frac{(\alpha_1 z+\beta_1)(z+4)+\alpha_2(z^2+1)}{(z^2+1)(z+4)}$$

$$=\frac{(\alpha_1+\alpha_2)z^2+(4\alpha_1+\beta_1)z+(4\beta_1+\alpha_2)}{(z^2+1)(z+4)}, \alpha_1+\alpha_2=3, 4\alpha_1+\beta_1=1, 4\beta_1+\alpha_2=2,$$

$$4(1-4\alpha_1)+3-\alpha_1=7-17\alpha_1=2, \alpha_1=\frac{5}{17}, \alpha_2=\frac{46}{17}, \beta_1=-\frac{3}{17}$$

$$f(z)=\frac{\alpha_1 z+\beta_1}{z^2(1+\frac{1}{z^2})}+\frac{\alpha_2}{4(1+\frac{z}{4})}=\frac{\alpha_1 z+\beta_1}{z^2}\sum_{n=0}^{\infty}(-\frac{1}{z^2})^n+\frac{\alpha_2}{4}\sum_{n=0}^{\infty}(-\frac{z}{4})^n$$

$$=\sum_{n=0}^{\infty}\alpha_1(-1)^n z^{-(2n+1)}+\sum_{n=0}^{\infty}\beta_1(-1)^n z^{-(2n+2)}+\frac{\alpha_2}{4}\sum_{n=0}^{\infty}(-\frac{1}{4})^n z^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{5}{17}(-1)^n z^{-(2n+1)}-\sum_{n=0}^{\infty}\frac{3}{17}(-1)^n z^{-(2n+2)}+\sum_{n=0}^{\infty}\frac{23}{34}(-4)^{-n} z^n$$

（別解）

$$f(z)=\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{\alpha_1}{z-i}+\frac{\alpha_2}{z+i}+\frac{\alpha_3}{z+4},$$

$$\alpha_1=\lim_{z\to i}(z-i)\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{3i^2+i+2}{2i(i+4)}=\frac{-3+i+2}{-2+8i}=\frac{-1+i}{-2+8i}=\frac{(-1+i)(-2-8i)}{4+64}=\frac{5+3i}{34}$$

$$\alpha_2=\lim_{z\to -i}(z+i)\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{3i^2+i+2}{2i(i+4)}=\frac{-3+i+2}{-2+8i}=\frac{-1+i}{-2+8i}=\frac{(-1+i)(-2-8i)}{4+64}=\frac{5-3i}{34}$$

$$\alpha_3=\lim_{z\to -4}(z+4)\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{48-4+2}{16+1}=\frac{46}{17}$$

$$f(z)=\frac{3z^2+z+2}{(z^2+1)(z+4)}=\frac{\alpha_1}{z-i}+\frac{\alpha_2}{z+i}+\frac{\alpha_3}{z+4}=\frac{\alpha_1}{z(1-\frac{i}{z})}+\frac{\alpha_2}{z(1+\frac{i}{z})}+\frac{\alpha_3}{4(1+\frac{z}{4})}$$

$$=\frac{\alpha_1}{z}\sum_{n=0}^{\infty}(\frac{i}{z})^n+\frac{\alpha_2}{z}\sum_{n=0}^{\infty}(\frac{-i}{z})^n+\frac{\alpha_3}{4}\sum_{n=0}^{\infty}(-\frac{z}{4})^n=\sum_{n=0}^{\infty}\{\alpha_1 i^n+\alpha_2(-i)^n\}z^{-(n+1)}+\sum_{n=0}^{\infty}\frac{\alpha_3}{4}(-4)^{-n}z^n,$$

$n+1=n'$

$$f(z)=\sum_{n'=1}^{\infty}\{\alpha_1 i^{n'-1}+\alpha_2(-i)^{n'-1}\}z^{-n'}+\sum_{n=0}^{\infty}\frac{\alpha_3}{4}(-4)^{-n}z^n, n'\to n$$

$$f(z)=\sum_{n=1}^{\infty}\{\frac{5+3i}{34}i^{n-1}+\frac{5-3i}{34}(-i)^{n-1}\}z^{-n}+\sum_{n=0}^{\infty}\frac{23}{34}(-4)^{-n}z^n$$