2017.2.3

情報科学科・応用生命システム工学科

数学Ⅲ　　期末試験　解答

.

1.(5点×2＝10点）

(1)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},u(x,y)=x^3-3xy^2+x^2-y^2,f(0)=0$$

$$\frac{\partial u}{\partial x}=3x^2-3y^2+2x=\frac{\partial v}{\partial y},v=\int(3x^2-3y^2+2x)dy=3x^2y-y^3+2xy+C_1(x),$$

$$\frac{\partial u}{\partial y}=-6xy-2y=-\frac{\partial v}{\partial x}=-\{6xy+2y+C_1'(x)\},C_1'(x)=0,C_1(x)=C$$

$v=3x^2y-y^3+2xy+C,v(0,0)=C=0$より，$v(x,y)=3x^2y-y^3+2xy$

$$f(x,y)=x^3-3xy^2+x^2-y^2+i(3x^2y-y^3+2xy),y\to 0,x\to z,f(z)=z^3+z^2$$

(2)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},u(x,y)=x+e^x\cos y,f(0)=1$$

$$\frac{\partial u}{\partial x}=1+e^x\cos y=\frac{\partial v}{\partial y}$$

$$v=\int(1+e^x\cos y)dy=y+e^x\sin y+C_1(x)$$

$$\frac{\partial u}{\partial y}=-e^x\sin y=-\frac{\partial v}{\partial x}=-e^x\sin y-C_1'(x),C_1'(x)=0,C_1(x)=C,$$

$$v=y+e^x\sin y+C,f(0)=u(0,0)+iv(0,0)=1+iC=1,C=0$$

$$v(x,y)=y+e^x\sin y,f(z)=(x+e^x\cos y)+i(y+e^x\sin y),$$

$y\to 0,x\to z$と置き換える　$f(z)=z+e^z$

2.(5点×2＝10点）

(1)

$$J=\oint_C\frac{z}{z^2+2z+2}dz,C:|z-i|=2,z^2+2z+2=0,z=-1\pm\sqrt{1-2}=-1\pm i$$

$$c_1=-1+i\in D,$$

$$J=2\pi i\operatorname*{Res}_{z=c_1}(\frac{z}{z^2+2z+2})dz=2\pi i\lim_{z\to c_1}\{\frac{(z-c_1)z}{z^2+2z+2}\}=2\pi i\lim_{z\to c_1}(\frac{2z-c_1}{2z+2})=\pi i\frac{c_1}{c_1+1}$$

$$=\pi(-1+i)$$

(2)

$$J=\oint_C \frac{1}{z^4+6z^2+5}dz, C:|z|=2, z^4+6z^2+5=(z^2+1)(z^2+5)=0, z=\pm i, z=\pm\sqrt{5}i$$

$$z=i=c_1\in D, z=-i=c_2\in D, N=2$$

$$J=2\pi i\sum_{j=1}^{N}\underset{z=c_j}{\mathrm{Re}\,s}(\frac{1}{z^4+6z^2+5})dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)}{z^4+6z^2+5}\}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}(\frac{1}{4z^3+12z})$$

$$=\pi i(\frac{1}{2c_1^3+6c_1}+\frac{1}{2c_2^3+6c_2})=\pi i(\frac{1}{2i^3+6i}+\frac{1}{2(-1)^3i^3-6i})=0$$

3.(10点×２＝20点）

(1)

$$J=\int_{-\infty}^{\infty}\frac{1}{x^2+2x+5}dx, f(z)=\frac{1}{z^2+2z+5}, Rf(\mathrm{Re}^{i\theta})\to 0(R\to\infty),(0\le\theta\le\pi)$$

$$z^2+2z+5=0, z=-1\pm\sqrt{1-5}=-1\pm 2i, c_1=-1+2i\in D$$

$$J=2\pi i\underset{z=c_1}{\mathrm{Re}\,s}(\frac{1}{z^2+2z+5})=2\pi i\lim_{z\to c_1}\frac{(z-c_1)}{z^2+2z+5}=2\pi i\lim_{z\to c_1}\frac{1}{2z+2}=\frac{\pi i}{c_1+1}=\frac{\pi}{2}$$

(2)

$$J=\int_{-\infty}^{\infty}\frac{1}{x^4+5x^2+4}dx, f(z)=\frac{1}{z^4+5z^2+4}, Rf(\mathrm{Re}^{i\theta})\to 0(R\to\infty),(0\le\theta\le\pi)$$

$$z^4+5z^2+4=0,(z^2+1)(z^2+4)=0, z=\pm i, z=\pm 2i, c_1=i\in D, c_2=2i\in D, N=2$$

$$J=2\pi i\sum_{j=1}^{N}\underset{z=c_j}{\mathrm{Re}\,s}(f(z))dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)}{z^4+5z^2+4}\}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{1}{4z^3+10z}\}=\pi i(\frac{1}{2c_1^3+5c_1}+\frac{1}{2c_2^3+5c_2})$$

$$=\pi i(\frac{1}{2i^3+5i}+\frac{1}{16i^3+10i})=\pi(\frac{1}{-2+5}+\frac{1}{-16+10})=\pi(\frac{1}{3}-\frac{1}{6})=\frac{\pi}{6}$$

4.(10点×2＝20点）

(1)

$$S=\int_{-\infty}^{\infty}\frac{\cos\pi x}{x^2+2x+3}dx, f(z)=\frac{1}{z^2+2z+3}, m=\pi>0, f(\mathrm{Re}^{i\theta})\to 0(R\to\infty),(0\le\theta\le\pi)$$

$$z^2+2z+3=0, z=-1\pm\sqrt{1-3}=-1\pm\sqrt{2}i, c_1=-1+\sqrt{2}i\in D$$

$$J=S+iT=2\pi i\underset{z=c_1}{\mathrm{Re}\,s}(\frac{e^{i\pi z}}{z^2+2z+3})=2\pi i\lim_{z\to c_1}\frac{(z-c_1)e^{i\pi z}}{z^2+2z+3}=2\pi i\lim_{z\to c_1}\frac{e^{i\pi z}+(z-c_1)e^{i\pi z}i\pi}{2z+2}=\frac{\pi ie^{i\pi c_1}}{c_1+1}$$

$$=\frac{\pi e^{i\pi(-1+\sqrt{2}i)}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\pi(-e^{-\sqrt{2}\pi})=-\frac{\sqrt{2}}{2}\pi e^{-\sqrt{2}\pi}, S=\mathrm{Re}(J)=-\frac{\sqrt{2}}{2}\pi e^{-\sqrt{2}\pi}$$

(2)

$$T=\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^4+1}dx, f(z)=\frac{z}{z^4+1}, m=\pi>0, f(\mathrm{R}e^{i\theta})\to 0(R\to\infty),(0\le\theta\le\pi)$$

$$z^4+1=0, z^4=-1=e^{\pi i+2n\pi i}, z=e^{\frac{\pi i+2n\pi i}{4}}, n=0,1,2,3, z_0=e^{\frac{\pi i}{4}}, z_1=e^{\frac{3\pi i}{4}}, z_2=e^{\frac{5\pi i}{4}}, z_3=e^{\frac{7\pi i}{4}},$$

$$c_1=e^{\frac{\pi i}{4}}\in D, c_2=e^{\frac{3\pi i}{4}}\in D, N=2$$

$$J=S+iT=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j}(\frac{ze^{i\pi z}}{z^4+1})dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{(z-c_j)ze^{i\pi z}}{z^4+1}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{(2z-c_j)e^{i\pi z}+(z-c_j)ze^{i\pi z}i\pi}{4z^3}$$

$$=2\pi i(\frac{c_1e^{i\pi c_1}}{4c_1^3}+\frac{c_2e^{i\pi c_2}}{4c_2^3})=\frac{\pi i}{2}(\frac{e^{i\pi c_1}}{c_1^2}+\frac{e^{i\pi c_2}}{c_2^2})=\frac{\pi i}{2}(\frac{e^{i\pi c_1}}{e^{\frac{\pi i}{2}}}+\frac{e^{i\pi c_2}}{e^{\frac{3\pi i}{2}}})=\frac{\pi i}{2}(\frac{e^{i\pi c_1}}{i}-\frac{e^{i\pi c_2}}{i})=\frac{\pi}{2}(e^{i\pi e^{\frac{\pi i}{4}}}-e^{i\pi e^{\frac{3\pi i}{4}}})$$

$$=\frac{\pi}{2}\{e^{i\pi(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})}-e^{i\pi(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})}\}=\frac{\pi}{2}(e^{i\pi\frac{\sqrt{2}}{2}}-e^{-i\pi\frac{\sqrt{2}}{2}})e^{-\frac{\sqrt{2}}{2}\pi}=\frac{i\pi}{2i}(e^{i\pi\frac{\sqrt{2}}{2}}-e^{-i\pi\frac{\sqrt{2}}{2}})e^{-\frac{\sqrt{2}}{2}\pi}=i\pi\sin(\frac{\sqrt{2}}{2}\pi)e^{-\frac{\sqrt{2}}{2}\pi}$$

$$T=\operatorname{Im}(J)=\pi\sin(\frac{\sqrt{2}}{2}\pi)e^{-\frac{\sqrt{2}}{2}\pi}$$

5.(10点×2=20点）

(1)

$$J=\int_0^{2\pi}\frac{1}{2+\cos\theta}d\theta, z=e^{i\theta}, dz=e^{i\theta}id\theta=zid\theta, J=\oint_{|z|=1}\frac{1}{2+\frac{z+z^{-1}}{2}}\frac{dz}{zi}=\oint_{|z|=1}\frac{2}{i}\frac{1}{z^2+4z+1}dz,$$

$$z^2+4z+1=0, z=-2\pm\sqrt{4-1}=-2\pm\sqrt{3}, c_1=-2+\sqrt{3},$$

$$J=2\pi i\operatorname*{Res}_{z=c_1}(\frac{2}{i}\frac{1}{z^2+4z+1})=4\pi\lim_{z\to c_1}\{\frac{(z-c_1)}{z^2+4z+1}\}=4\pi\lim_{z\to c_1}(\frac{1}{2z+4})=\frac{2\pi}{c_1+2}=\frac{2}{\sqrt{3}}\pi=\frac{2\sqrt{3}}{3}\pi$$

(2)

$$J=\int_0^{2\pi}\frac{4+\cos\theta}{3+\sin\theta}d\theta, z=e^{i\theta}, dz=e^{i\theta}id\theta=zid\theta, J=\oint_{|z|=1}\frac{4+\frac{z+z^{-1}}{2}}{3+\frac{z-z^{-1}}{2i}}\frac{dz}{zi}=\oint_{|z|=1}\frac{8i+iz+iz^{-1}}{6i+z-z^{-1}}\times\frac{1}{zi}dz$$

$$=\oint_{|z|=1}\frac{8z+z^2+1}{(z^2+6iz-1)z}dz=\oint_{|z|=1}\frac{z^2+8z+1}{z(z^2+6iz-1)}dz, z^2+6zi-1=0, z=-3i\pm\sqrt{-9+1}=-3i\pm 2\sqrt{2}i$$

$c_1=0\in D, c_2=(-3+2\sqrt{2})i\in D, N=2$

$$J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j}s\{\frac{z^2+8z+1}{z(z^2+6zi-1)}\}=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z-c_j)(z^2+8z+1)}{z^3+6iz^2-z}\}$$

$$=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\{\frac{(z^2+8z+1)+(z-c_j)(2z+8)}{3z^2+12iz-1}\}=\pi i\sum_{j=1}^{N}\lim_{z\to c_j}(\frac{z^2+8z+1}{3z^2+12iz-1}), c_2^2=-6c_2i+1$$

$$J=\pi i(-1+\frac{c_2^2+8c_2+1}{3c_2^2+12ic_2-1})=\pi i(-1+\frac{-6ic_2+1+8c_2+1}{-18c_2i+3+12ic_2-1})=\pi i(-1+\frac{-6ic_2+1+8c_2+1}{-6c_2i+2})$$

$$=\pi i(\frac{6c_2i-2-6ic_2+1+8c_2+1}{-6c_2i+2})=\pi i\frac{8c_2}{-6c_2i+2}=\frac{4c_2i}{1-3c_2i}\pi=\frac{-4(-3+2\sqrt{2})}{1+3(-3+2\sqrt{2})}\pi=\frac{4(3-2\sqrt{2})}{6\sqrt{2}-8}\pi$$

$$=\frac{2(3-2\sqrt{2})}{(3\sqrt{2}-4)}\pi=\frac{2(3-2\sqrt{2})}{\sqrt{2}(3-2\sqrt{2})}\pi=\sqrt{2}\pi$$

6.(10点×2＝20点）

(1)

$$f(z)=\frac{z+2}{z^2+8z+7}, D=\{z\big|2<|z|<5\},$$

$$f(z)=\frac{z+2}{(z+1)(z+7)}=\frac{\alpha_1}{z+1}+\frac{\alpha_2}{z+7}, \alpha_1=\lim_{z\to-1}\frac{(z+1)(z+2)}{(z+1)(z+7)}=\frac{1}{6}, \alpha_2=\lim_{z\to-7}\frac{(z+7)(z+2)}{(z+1)(z+7)}=\frac{5}{6}$$

$$f(z)=\frac{\alpha_1}{z(1+\frac{1}{z})}+\frac{\alpha_2}{7(1+\frac{z}{7})}=\frac{\alpha_1}{z}\sum_{n=0}^{\infty}(-\frac{1}{z})^n+\frac{\alpha_2}{7}\sum_{n=0}^{\infty}(-\frac{z}{7})^n=\sum_{n=0}^{\infty}\frac{1}{6}(-1)^n z^{-(n+1)}+\sum_{n=0}^{\infty}\frac{5}{42}(-\frac{1}{7})^n z^n, n+1=n'$$

$$f(z)=\sum_{n'=1}^{\infty}\frac{1}{6}(-1)^{n'-1}z^{-n'}+\sum_{n=0}^{\infty}\frac{5}{42}(-\frac{1}{7})^n z^n=\sum_{n=1}^{\infty}\frac{1}{6}(-1)^{n-1}z^{-n}+\sum_{n=0}^{\infty}\frac{5}{42}(-7)^{-n}z^n$$

(2)

$$f(z)=\frac{z^2+z+2}{(z^2+2)(z+5)},D=\{z\big|3<|z|<4\},$$

$$f(z)=\frac{z^2+z+2}{(z^2+2)(z+5)}=\frac{\alpha_1 z+\beta_1}{z^2+2}+\frac{\alpha_2}{z+5}=\frac{(\alpha_1 z+\beta_1)(z+5)+\alpha_2(z^2+2)}{(z^2+2)(z+5)}$$

$$=\frac{(\alpha_1+\alpha_2)z^2+(5\alpha_1+\beta_1)z+(5\beta_1+2\alpha_2)}{(z^2+2)(z+5)},\alpha_1+\alpha_2=1,5\alpha_1+\beta_1=1,5\beta_1+2\alpha_2=2,$$

$$5(1-5\alpha_1)+2(1-\alpha_1)=7-27\alpha_1=2,\alpha_1=\frac{5}{27},\alpha_2=\frac{22}{27},\beta_1=1-\frac{25}{27}=\frac{2}{27}$$

$$f(z)=\frac{\alpha_1 z+\beta_1}{z^2(1+\frac{2}{z^2})}+\frac{\alpha_2}{5(1+\frac{z}{5})}=\frac{\alpha_1 z+\beta_1}{z^2}\sum_{n=0}^{\infty}(-\frac{2}{z^2})^n+\frac{\alpha_2}{5}\sum_{n=0}^{\infty}(-\frac{z}{5})^n$$

$$=\sum_{n=0}^{\infty}\alpha_1(-2)^n z^{-(2n+1)}+\sum_{n=0}^{\infty}\beta_1(-2)^n z^{-(2n+2)}+\frac{\alpha_2}{5}\sum_{n=0}^{\infty}(-5)^{-n}z^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{5}{27}(-2)^n z^{-(2n+1)}+\sum_{n=0}^{\infty}\frac{2}{27}(-2)^n z^{-(2n+2)}-\sum_{n=0}^{\infty}\frac{22}{27}(-5)^{-n-1}z^n$$

(别解)

$$f(z)=\frac{z^2+z+2}{(z^2+2)(z+5)}=\frac{\alpha_1}{z-\sqrt{2}i}+\frac{\alpha_2}{z+\sqrt{2}i}+\frac{\alpha_3}{z+5}$$

$$\alpha_1=\lim_{z\to\sqrt{2}i}\frac{(z-\sqrt{2}i)(z^2+z+2)}{(z^2+2)(z+5)}=\lim_{z\to\sqrt{2}i}\frac{(z^2+z+2)}{(z+\sqrt{2}i)(z+5)}=\frac{5-\sqrt{2}i}{54}$$

$$\alpha_2=\lim_{z\to-\sqrt{2}i}\frac{(z+\sqrt{2}i)(z^2+z+2)}{(z^2+2)(z+5)}=\lim_{z\to-\sqrt{2}i}\frac{(z^2+z+2)}{(z-\sqrt{2}i)(z+5)}=\frac{5+\sqrt{2}i}{54}$$

$$\alpha_3=\lim_{z\to-5}\frac{(z+5)(z^2+z+2)}{(z^2+2)(z+5)}=\lim_{z\to-5}\frac{(z^2+z+2)}{(z^2+2)}=\frac{22}{27}$$

$$f(z)=\frac{\alpha_1}{z(1-\frac{\sqrt{2}i}{z})}+\frac{\alpha_2}{z(1+\frac{\sqrt{2}i}{z})}+\frac{\alpha_3}{5(1+\frac{z}{5})}=\sum_{n=0}^{\infty}\frac{\alpha_1}{z}(\frac{\sqrt{2}i}{z})^n+\sum_{n=0}^{\infty}\frac{\alpha_2}{z}(-\frac{\sqrt{2}i}{z})^n+\sum_{n=0}^{\infty}\frac{\alpha_3}{5}(-\frac{z}{5})^n$$

$$f(z)=\sum_{n=1}^{\infty}\{\frac{5-\sqrt{2}i}{54}(\sqrt{2}i)^{n-1}+\frac{5+\sqrt{2}i}{54}(-\sqrt{2}i)^{n-1}\}z^{-n}-\sum_{n=0}^{\infty}\frac{22}{27}(-5)^{-n-1}z^n$$