数学Ⅲ　レポート　No.1　　　解答　　　　　　(2016)

1.　(5 点×5=25 点)

(1) $e^{\frac{\pi}{3}i}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(2) $4e^{\frac{\pi}{6}i}=4(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=4(\frac{\sqrt{3}}{2}+\frac{1}{2}i)=2\sqrt{3}+2i$

(3) $5e^{\frac{5\pi}{3}i}=5e^{\frac{(6-1)\pi}{3}i}=5e^{2\pi i-\frac{\pi}{3}i}=5e^{-\frac{\pi}{3}i}=5(\cos\frac{\pi}{3}-\sin\frac{\pi}{3}i)=5(\frac{1}{2}-\frac{\sqrt{3}}{2}i)=\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

(4)

$$2e^{\frac{5\pi}{12}i}=2e^{(\frac{1}{12})5\pi i}=2e^{(\frac{1}{3}-\frac{1}{4})5\pi i}=2e^{\frac{5}{3}\pi i-\frac{5}{4}\pi i}=2e^{2\pi i-\frac{\pi}{3}i-\pi i-\frac{1}{4}\pi i}=2e^{\pi i-\frac{\pi}{3}i-\frac{\pi}{4}i}=2e^{\pi i}e^{-\frac{\pi}{3}i}e^{-\frac{\pi}{4}i}$$

$$=2(\cos\pi+\sin\pi i)(\cos\frac{\pi}{3}-\sin\frac{\pi}{3}i)(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})=2(-1)(\frac{1}{2}-\frac{\sqrt{3}}{2}i)(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)$$

$$=-\frac{\sqrt{2}}{2}(1-\sqrt{3}i)(1-i)=-\frac{\sqrt{2}}{2}\{(1-\sqrt{3})-(1+\sqrt{3})i\}=\frac{\sqrt{6}-\sqrt{2}}{2}+\frac{\sqrt{6}+\sqrt{2}}{2}i$$

(5)

$$8e^{\frac{\pi}{8}i}=8(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}i)$$

$$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=2\cos^2\frac{\theta}{2}-1=1-2\sin^2\frac{\theta}{2}$$

$$\cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2},\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}},\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\pi}{8}=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\frac{\pi}{8}=\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$8e^{\frac{\pi}{8}i}=8(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8})=4(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}})$$

2.　(5 点×5=25 点)

(1) $1-i=\sqrt{1+1}e^{-\frac{\pi}{4}i}=\sqrt{2}e^{-\frac{\pi}{4}i}$

(2) $\sqrt{3}+i=\sqrt{4}e^{\frac{\pi}{6}i}=2e^{\frac{\pi}{6}i}$

(3) $(1-\sqrt{3}i)^3=(\sqrt{1+3}e^{-\frac{\pi}{3}i})^3=(2e^{-\frac{\pi}{3}i})^3=8e^{-\pi i}=8e^{\pi i}$

(4) $\dfrac{(\sqrt{3}+i)^5}{8}=\dfrac{1}{8}(2e^{\frac{\pi}{6}i})^5=\dfrac{2^5}{8}e^{\frac{5\pi}{6}i}=4e^{\frac{5\pi}{6}i}$

(5) $\dfrac{1+i}{2}(\sqrt{3}-i)=\dfrac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}\sqrt{3+1}e^{-\frac{\pi}{6}i}=\sqrt{2}e^{\frac{\pi}{12}i}$

3. (10 点×2=20 点)

(1)

$$z^3=8=2^3e^{2\pi ki},z=2e^{\frac{2\pi ki}{3}},(k=0,1,2)$$

$$z_0=2,z_1=2e^{\frac{2\pi}{3}i}=2(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)=-1+\sqrt{3}i,z_2=2e^{\frac{4\pi}{3}i}=2(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)=-1-\sqrt{3}i$$

(2)

$$z^4=-4=4e^{\pi i}=4e^{\pi i+2\pi ki},z=4^{\frac{1}{4}}e^{\frac{\pi+2\pi k}{4}i},(k=0,1,2,3)$$

$$z_0=\sqrt{2}e^{\frac{\pi}{4}i}=\sqrt{2}(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)=1+i,z_1=\sqrt{2}e^{\frac{3\pi}{4}i}=\sqrt{2}(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)=-1+i,$$

$$z_2=\sqrt{2}e^{\frac{5\pi}{4}i}=\sqrt{2}(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)=-1-i,z_4=\sqrt{2}e^{\frac{7\pi}{4}i}=\sqrt{2}(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)=1-i$$

4. (10 点×3=30 点)

(1)

$$|z+a|+|z-a|=\ell$$

$$z=x+iy$$

とおく．

$$|x+iy+a|+|x+iy-a|=\ell$$

$$\sqrt{(x+a)^2+y^2}+\sqrt{(x-a)^2+y^2}=\ell$$

両辺を２乗する．

$$(x+a)^2+y^2+(x-a)^2+y^2+2\sqrt{(x^2+y^2+a^2+2ax)(x^2+y^2+a^2-2ax)}=\ell^2$$

$$2\sqrt{(x^2+y^2+a^2)^2-4a^2x^2}=\ell^2-2(x^2+y^2+a^2)$$

両辺を２乗する．

$$4\{(x^2+y^2+a^2)^2-4a^2x^2\}=\ell^4-4\ell^2(x^2+y^2+a^2)+4(x^2+y^2+a^2)^2$$

$$-16a^2x^2=\ell^4-4\ell^2(x^2+y^2+a^2)$$

$$(4\ell^2-16a^2)x^2+4\ell^2y^2=\ell^4-4\ell^2a^2, 4(\ell^2-4a^2)x^2+4\ell^2y^2=\ell^2(\ell^2-4a^2)$$

$$\frac{4}{\ell^2}x^2+\frac{4}{\ell^2-4a^2}y^2=1, \frac{x^2}{(\frac{\ell}{2})^2}+\frac{y^2}{(\frac{\sqrt{\ell^2-4a^2}}{2})^2}=1$$

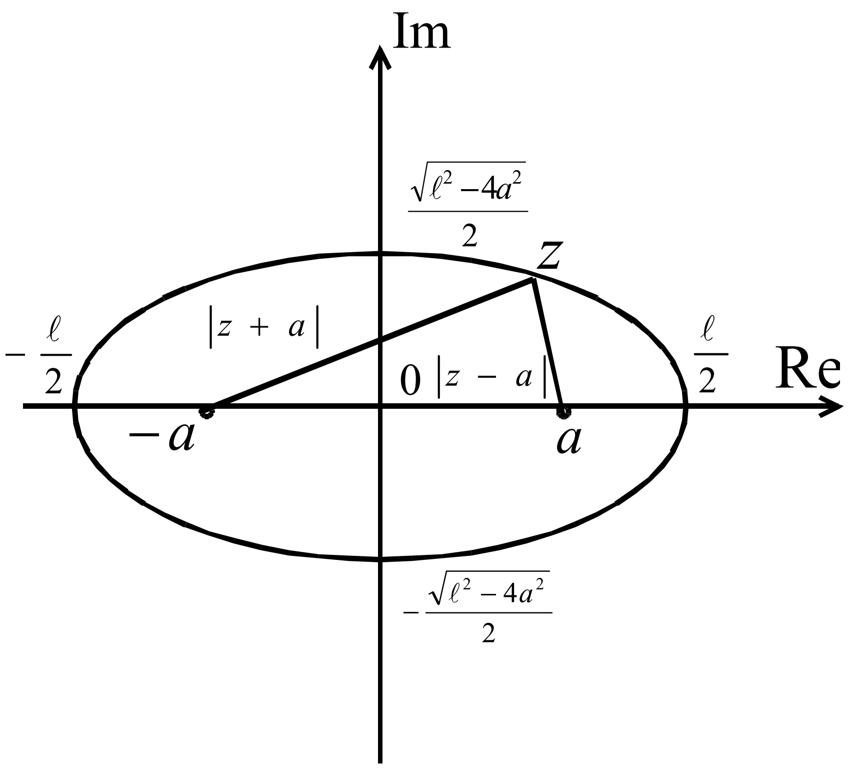

Fig.1　4(1)　楕円 (長径ℓ、短径$\sqrt{\ell^2-4a^2}$)

(2)

$$|z+a|-|z-a|=\ell$$

$z=x+iy$

とおく.

$$|x+iy+a|-|x+iy-a|=\ell$$

$$\sqrt{(x+a)^2+y^2}-\sqrt{(x-a)^2+y^2}=\ell$$

両辺を２乗する.

$$(x+a)^2+y^2+(x-a)^2+y^2-2\sqrt{(x^2+y^2+a^2+2ax)(x^2+y^2+a^2-2ax)}=\ell^2$$

$$2(x^2+y^2+a^2)-\ell^2=2\sqrt{(x^2+y^2+a^2)^2-4a^2x^2}$$

両辺を２乗する.

$$4(x^2+y^2+a^2)^2-4\ell^2(x^2+y^2+a^2)+\ell^4=4(x^2+y^2+a^2)^2-16a^2x^2$$

$$\ell^4-4\ell^2a^2=(4\ell^2-16a^2)x^2+4\ell^2y^2$$

$$4(4a^2-\ell^2)x^2-4\ell^2y^2=\ell^2(4a^2-\ell^2),2a>\ell>0$$

$$\frac{4}{\ell^2}x^2-\frac{4}{4a^2-\ell^2}y^2=1,\frac{x^2}{(\frac{\ell}{2})^2}-\frac{y^2}{(\frac{\sqrt{4a^2-\ell^2}}{2})^2}=1$$

$$|z+a|>|z-a|$$

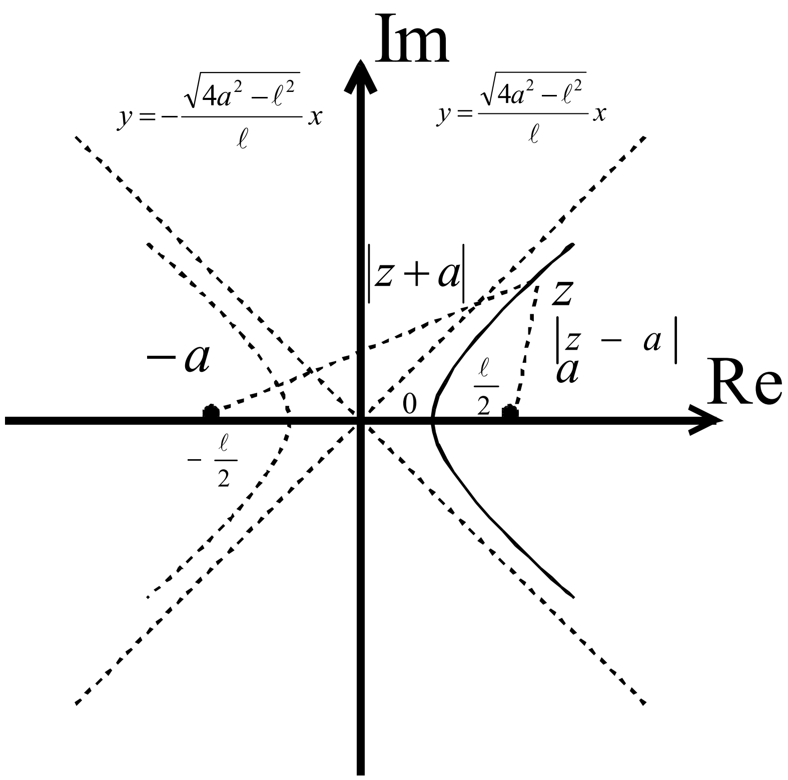

Fig.2　双曲線　　ただし x>0 の範囲

(3)

$$\mathrm{Re}(z+a)=|z-a|, z=x+yi, \mathrm{Re}(x+yi+a)=|x+yi-a|, x+a=\sqrt{(x-a)^2+y^2}$$

$$(x+a)^2=(x-a)^2+y^2, x^2+2ax+a^2=x^2-2ax+a^2+y^2, 4ax=y^2, x=\frac{1}{4a}y^2$$

焦点(a,0),準線 x=-a の放物線

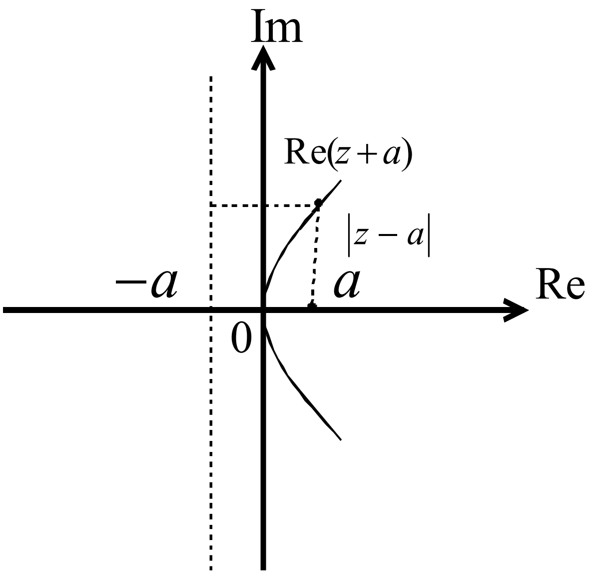

Fig.3　　放物線