数学Ⅲ　レポート　No.3　　解答　　　　　　(2016)

1.　(5 点×5=25 点)

(1)

$$f(z)=\tan z, f'(z)=\sec^2 z=(\cos z)^{-2}, f''(z)=(-2)(\cos z)^{-3}(-\sin z)=2(\cos z)^{-3}(\sin z),$$

$$f'''(z)=2\times(-3)(\cos z)^{-4}\times(-\sin z)\times\sin z+2(\cos z)^{-3}\cos z$$

$$f(0)=0, f'(0)=1, f''(0)=0, f'''(0)=2$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=z+\frac{2}{3!}z^3+\sim=z+\frac{1}{3}z^3+\sim$$

(2)

$$f(z)=e^{z\sin z}, f'(z)=e^{z\sin z}(\sin z+z\cos z),$$

$$f''(z)=e^{z\sin z}(\sin z+z\cos z)^2+e^{z\sin z}(2\cos z-z\sin z)=e^{z\sin z}\{(\sin z+z\cos z)^2+(2\cos z-z\sin z)\},$$

$$f'''(z)=e^{z\sin z}(\sin z+z\cos z)\{(\sin z+z\cos z)^2+(2\cos z-z\sin z)\}$$

$$+e^{z\sin z}\{2(\sin z+z\cos z)(2\cos z-z\sin z)-(3\sin z+z\cos z)\}$$

$$f(0)=1, f'(0)=0, f''(0)=2, f'''(0)=0$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=1+\frac{2}{2!}z^2+\sim=1+z^2+\sim$$

(3)

$$f(z)=\log(1+z+z^2), f'(z)=\frac{1+2z}{1+z+z^2}=(1+2z)(1+z+z^2)^{-1},$$

$$f''(z)=2(1+z+z^2)^{-1}-(1+2z)^2(1+z+z^2)^{-2},$$

$$f'''(z)=-2(1+2z)(1+z+z^2)^{-2}-4(1+2z)(1+z+z^2)^{-2}+2(1+2z)^3(1+z+z^2)^{-3}$$

$$f(0)=0, f'(0)=1, f''(0)=1, f'''(0)=-4$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=z+\frac{1}{2!}z^2-\frac{4}{3!}z^3\sim=z+\frac{1}{2}z^2-\frac{2}{3}z^3+\sim$$

(4)

$$f(z)=\sin(\pi+\log(1+z)), f'(z)=\cos(\pi+\log(1+z))\frac{1}{1+z}=\cos(\pi+\log(1+z))(1+z)^{-1},$$

$$f''(z)=-\sin(\pi+\log(1+z))(1+z)^{-2}-\cos(\pi+\log(1+z))(1+z)^{-2}$$

$$=-\{\sin(\pi+\log(1+z))+\cos(\pi+\log(1+z))\}(1+z)^{-2},$$

$$f'''(z)=\{\cos(\pi+\log(1+z))+3\sin(\pi+\log(1+z))\}(1+z)^{-3}$$

$$f(0)=0, f'(0)=-1, f''(0)=1, f'''(0)=-1$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=-z+\frac{1}{2}z^2-\frac{1}{6}z^3+\sim$$

(5)

$$f(z)=\frac{z}{e^z+1}, f'(z)=(e^z+1)^{-2}\{1+(1-z)e^z\}, f''(z)=(e^z+1)^{-3}e^z(-2-2e^z+ze^z-z),$$

$$f'''(z)=-3(e^z+1)^{-4}e^{2z}(-2-2e^z+ze^z-z)+(e^z+1)^{-3}e^z(-2-2e^z+ze^z-z)+(e^z+1)^{-3}e^z(-1-e^z+ze^z)$$

$$f(0)=0, f'(0)=\frac{1}{2}, f''(0)=-\frac{1}{2}, f'''(0)=0$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=\frac{1}{2}z-\frac{1}{4}z^2+\sim$$

2. (5 点×5=25 点)

(1)

$$f(z)=\frac{1}{1-\alpha z}=\sum_{n=0}^{\infty}\alpha^n z^n, f(z)=\frac{1}{1-(2+3i)z}=\sum_{n=0}^{\infty}(2+3i)^n z^n,$$

$$\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(2+3i)^n}{(2+3i)^{n+1}}\right|=\frac{1}{|2+3i|}=\frac{1}{\sqrt{4+9}}=\frac{1}{\sqrt{13}}$$

(2)

$$f(z)=\frac{1}{1-(1+i)z}+\frac{1}{1-(2-i)z}=\sum_{n=0}^{\infty}(1+i)^n z^n+\sum_{n=0}^{\infty}(2-i)^n z^n=\sum_{n=0}^{\infty}\{(1+i)^n+(2-i)^n\}z^n$$

$$\rho_1=\lim_{n\to\infty}\left|\frac{(1+i)^n}{(1+i)^{n+1}}\right|=\lim_{n\to\infty}\frac{1}{|1+i|}=\frac{1}{\sqrt{2}}, \rho_2=\lim_{n\to\infty}\left|\frac{(2-i)^n}{(2-i)^{n+1}}\right|=\lim_{n\to\infty}\frac{1}{|2-i|}=\frac{1}{\sqrt{5}}$$

$$\rho=\min(\rho_1,\rho_2)=\min(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{5}})=\frac{1}{\sqrt{5}}$$

(3)

$$f(z)=\frac{1}{\{1-(1+3i)z\}\{1-(2+4i)z\}}=\frac{1}{(1-\alpha_1 z)(1-\alpha_2 z)}=\frac{\beta_1}{1-\alpha_1 z}+\frac{\beta_2}{1-\alpha_2 z}=\frac{(\beta_1+\beta_2)-(\beta_1\alpha_2+\beta_2\alpha_1)z}{(1-\alpha_1 z)(1-\alpha_2 z)},$$

$$\beta_1+\beta_2=1,\beta_1\alpha_2+\beta_2\alpha_1=0,\beta_1=\frac{\alpha_1}{\alpha_1-\alpha_2}=-\frac{1+3i}{1+i},\beta_2=-\frac{\alpha_2}{\alpha_1-\alpha_2}=\frac{2+4i}{1+i}$$

$$f(z)=\frac{\beta_1}{1-\alpha_1 z}+\frac{\beta_2}{1-\alpha_2 z}=\frac{-\frac{1+3i}{1+i}}{1-(1+3i)z}+\frac{\frac{2+4i}{1+i}}{1-(2+4i)z}=-\frac{1+3i}{1+i}\sum_{n=0}^{\infty}(1+3i)^n z^n+\frac{2+4i}{1+i}\sum_{n=0}^{\infty}(2+4i)^n z^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{1+i}\{-(1+3i)^{n+1}+(2+4i)^{n+1}\}z^n$$

$$\rho_1=\lim_{n\to\infty}\left|\frac{-(1+3i)^{n+1}}{-(1+3i)^{n+2}}\right|=\frac{1}{|1+3i|}=\frac{1}{\sqrt{10}},\rho_2=\lim_{n\to\infty}\left|\frac{(2+4i)^{n+1}}{(2+4i)^{n+2}}\right|=\frac{1}{|2+4i|}=\frac{1}{\sqrt{20}}$$

$$\rho=\min(\rho_1,\rho_2)=\min(\frac{1}{\sqrt{10}},\frac{1}{\sqrt{20}})=\frac{1}{\sqrt{20}}$$

(4)

$$f(z)=\frac{1}{1+z+z^2}=\frac{1}{(z-\lambda_1)(z-\lambda_2)}=\frac{\beta_1}{z-\lambda_1}+\frac{\beta_2}{z-\lambda_2}$$

$$z^2+z+1=0,\lambda_1=\frac{-1+\sqrt{3}i}{2},\lambda_2=\frac{-1-\sqrt{3}i}{2},\frac{1}{\lambda_1}=\lambda_2,\frac{1}{\lambda_2}=\lambda_1$$

$$\beta_1=\lim_{z\to\lambda_1}(z-\lambda_1)\times\frac{1}{(z-\lambda_1)(z-\lambda_2)}=\frac{1}{\lambda_1-\lambda_2}=\frac{1}{\sqrt{3}i},\beta_2=\lim_{z\to\lambda_2}(z-\lambda_2)\times\frac{1}{(z-\lambda_1)(z-\lambda_2)}=\frac{1}{\lambda_2-\lambda_1}=-\frac{1}{\sqrt{3}i}$$

$$f(z)=\frac{-\frac{\beta_1}{\lambda_1}}{1-\frac{z}{\lambda_1}}+\frac{-\frac{\beta_2}{\lambda_2}}{1-\frac{z}{\lambda_2}}=\sum_{n=0}^{\infty}\{-\frac{\beta_1}{\lambda_1}(\frac{z}{\lambda_1})^n-\frac{\beta_2}{\lambda_2}(\frac{z}{\lambda_2})^n\}=\sum_{n=0}^{\infty}(-\beta_1\lambda_2^{n+1}-\beta_2\lambda_1^{n+1})z^n$$

$$=\sum_{n=0}^{\infty}\{-\frac{1}{\sqrt{3}i}(\frac{-1-\sqrt{3}i}{2})^{n+1}+\frac{1}{\sqrt{3}i}(\frac{-1+\sqrt{3}i}{2})^{n+1}\}z^n=\sum_{n=0}^{\infty}\{\frac{1}{\sqrt{3}}i(\frac{-1-\sqrt{3}i}{2})^{n+1}-\frac{1}{\sqrt{3}}i(\frac{-1+\sqrt{3}i}{2})^{n+1}\}z^n$$

$$=\sum_{n=0}^{\infty}(a_n+b_n)z^n,a_n=\frac{1}{\sqrt{3}}i(\frac{-1-\sqrt{3}i}{2})^{n+1},b_n=-\frac{1}{\sqrt{3}}i(\frac{-1+\sqrt{3}i}{2})^{n+1}$$

$$\rho_1=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\frac{1}{\left|\frac{-1-\sqrt{3}i}{2}\right|}=\frac{2}{\sqrt{1+3}}=1,\rho_2=\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|=\frac{1}{\left|\frac{-1+\sqrt{3}i}{2}\right|}=\frac{2}{\sqrt{1+3}}=1$$

$$\rho=\min(\rho_1,\rho_2)=\min(1,1)=1$$

(5)

$$f(z)=\frac{4+3z}{1+2z+5z^2}=\frac{\frac{3}{5}z+\frac{4}{5}}{(z-\lambda_1)(z-\lambda_2)}=\frac{\beta_1}{z-\lambda_1}+\frac{\beta_2}{z-\lambda_2}$$

$$5z^2+2z+1=0,\lambda_1=\frac{-1+2i}{5},\lambda_2=\frac{-1-2i}{5},\frac{1}{\lambda_1}=5\lambda_2,\frac{1}{\lambda_2}=5\lambda_1$$

$$\beta_1=\lim_{z\to\lambda_1}(z-\lambda_1)\times\frac{\frac{3}{5}z+\frac{4}{5}}{(z-\lambda_1)(z-\lambda_2)}=\frac{\frac{3}{5}\lambda_1+\frac{4}{5}}{\lambda_1-\lambda_2}=\frac{6-17i}{20},$$

$$\beta_2=\lim_{z\to\lambda_2}(z-\lambda_2)\times\frac{\frac{3}{5}z+\frac{4}{5}}{(z-\lambda_1)(z-\lambda_2)}=\frac{\frac{3}{5}\lambda_2+\frac{4}{5}}{\lambda_2-\lambda_1}=\frac{6+17i}{20}$$

$$f(z)=\frac{-\frac{\beta_1}{\lambda_1}}{1-\frac{z}{\lambda_1}}+\frac{-\frac{\beta_2}{\lambda_2}}{1-\frac{z}{\lambda_2}}=\sum_{n=0}^{\infty}\{-\frac{\beta_1}{\lambda_1}(\frac{z}{\lambda_1})^n-\frac{\beta_2}{\lambda_2}(\frac{z}{\lambda_2})^n\}=\sum_{n=0}^{\infty}\{-\beta_1(5\lambda_2)^{n+1}-\beta_2(5\lambda_1)^{n+1}\}z^n$$

$$=\sum_{n=0}^{\infty}\{-\frac{6-17i}{20}(-1-2i)^{n+1}-\frac{6+17i}{20}(-1+2i)^{n+1}\}z^n$$

$$=\sum_{n=0}^{\infty}(a_n+b_n)z^n,a_n=-\frac{6-17i}{20}(-1-2i)^{n+1},b_n=-\frac{6+17i}{20}(-1+2i)^{n+1}$$

$$\rho_1=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\frac{1}{\left|-1-2i\right|}=\frac{1}{\sqrt{1+4}}=\frac{1}{\sqrt{5}},\rho_2=\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|=\frac{1}{\left|-1+2i\right|}=\frac{1}{\sqrt{1+4}}=\frac{1}{\sqrt{5}}$$

$$\rho=\min(\rho_1,\rho_2)=\min(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}})=\frac{1}{\sqrt{5}}$$

3. (5 点×10=50 点)

(1)

$$\log(-\sqrt{3}-i)=\log(2e^{-\frac{5}{6}\pi i+2n\pi i})=\log 2-\frac{5}{6}\pi i+2n\pi i=\log 2+(-\frac{5}{6}\pi+2n\pi)i,\qquad n\in Z$$

(2)

$$\log(i)=\log(e^{\frac{\pi}{2}i+2n\pi i})=(\frac{\pi}{2}+2n\pi)i,\qquad n\in Z$$

(3)

$$i^{3+i}=(e^{\frac{\pi}{2}i+2n\pi i})^{(3+i)}=e^{(\frac{\pi}{2}+2n\pi)3i+(\frac{\pi}{2}+2n\pi)i^2}=e^{\frac{3\pi}{2}i+6n\pi i-(\frac{\pi}{2}+2n\pi)}=e^{\frac{3\pi}{2}i}e^{6n\pi i}e^{-(\frac{\pi}{2}+2n\pi)}=-ie^{-(\frac{\pi}{2}+2n\pi)}$$

(4)

$$\cos(\frac{\pi}{4}+3i)=\cos\frac{\pi}{4}\cos 3i-\sin\frac{\pi}{4}\sin 3i=\frac{\sqrt{2}}{2}(\frac{e^{i\theta}+e^{-i\theta}}{2})_{\theta=3i}-\frac{\sqrt{2}}{2}(\frac{e^{i\theta}-e^{-i\theta}}{2i})_{\theta=3i}$$
$$=\frac{\sqrt{2}}{2}(\frac{e^{3i^2}+e^{-3i^2}}{2})-\frac{\sqrt{2}}{2}(\frac{e^{3i^2}-e^{-3i^2}}{2i})=\frac{\sqrt{2}}{4}(e^{-3}+e^{3})+\frac{\sqrt{2}}{4}(e^{-3}-e^{3})i$$

(5)

$$\sin(\frac{\pi}{3}+2i)=\sin\frac{\pi}{3}\cos 2i+\cos\frac{\pi}{3}\sin 2i=\frac{\sqrt{3}}{2}(\frac{e^{i\theta}+e^{-i\theta}}{2})_{\theta=2i}+\frac{1}{2}(\frac{e^{i\theta}-e^{-i\theta}}{2i})_{\theta=2i}$$
$$=\frac{\sqrt{3}}{2}(\frac{e^{2i^2}+e^{-2i^2}}{2})+\frac{1}{2}(\frac{e^{2i^2}-e^{-2i^2}}{2i})=\frac{\sqrt{3}}{4}(e^{-2}+e^{2})+\frac{1}{4}(e^{2}-e^{-2})i$$

(6)

$$\tan(\frac{\pi}{6}+4i)=\frac{\sin(\frac{\pi}{6}+4i)}{\cos(\frac{\pi}{6}+4i)}=(\frac{\frac{e^{i\theta}-e^{-i\theta}}{2i}}{\frac{e^{i\theta}+e^{-i\theta}}{2}})_{\theta=\frac{\pi}{6}+4i}=\frac{e^{i(\frac{\pi}{6}+4i)}-e^{-i(\frac{\pi}{6}+4i)}}{i\{e^{i(\frac{\pi}{6}+4i)}+e^{-i(\frac{\pi}{6}+4i)}\}}=\frac{e^{\frac{\pi}{6}i-4}-e^{-\frac{\pi}{6}i+4}}{i(e^{\frac{\pi}{6}i-4}+e^{-\frac{\pi}{6}i+4})}=\frac{e^{\frac{\pi}{3}i}-e^{8}}{i(e^{\frac{\pi}{3}i}+e^{8})}$$
$$=\frac{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}-e^{8}}{i(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}+e^{8})}=\frac{\frac{1}{2}+i\frac{\sqrt{3}}{2}-e^{8}}{i(\frac{1}{2}+i\frac{\sqrt{3}}{2}+e^{8})}=\frac{(\frac{1}{2}-e^{8}+i\frac{\sqrt{3}}{2})(\frac{1}{2}+e^{8}-i\frac{\sqrt{3}}{2})}{i\{(\frac{1}{2}+e^{8})^2+\frac{3}{4}\}}=\frac{(\frac{1}{2}-e^{8})(\frac{1}{2}+e^{8})+i\sqrt{3}e^{8}+\frac{3}{4}}{i\{(\frac{1}{2}+e^{8})^2+\frac{3}{4}\}}$$
$$=\frac{1-e^{16}+i\sqrt{3}e^{8}}{i\{(\frac{1}{2}+e^{8})^2+\frac{3}{4}\}}=\frac{\sqrt{3}e^{8}+i(e^{16}-1)}{1+e^{8}+e^{16}}=\frac{\sqrt{3}+(e^{8}-e^{-8})i}{e^{8}+e^{-8}+1}$$

(7)

$$\log(1+i)=\log(\sqrt{2}e^{\frac{\pi}{4}i+2n\pi i})=\frac{1}{2}\log 2+(\frac{\pi}{4}+2n\pi)i, \qquad n\in Z$$

(8)

$$\arcsin(2i)=y,$$
$$\arcsin z=y, z=\sin y=\frac{e^{iy}-e^{-yi}}{2i}, X=e^{iy}$$
$$z=\frac{X-X^{-1}}{2i}, X^2-2izX-1=0, X=iz\pm\sqrt{1-z^2}, z=2i$$
$$e^{iy}=2i^2\pm\sqrt{1+4}=-2\pm\sqrt{5}$$
$$e^{iy}=-2+\sqrt{5}=(\sqrt{5}-2)e^{0i}=(\sqrt{5}-2)e^{2n\pi i}, iy=\log(\sqrt{5}-2)+2n\pi i, y=2n\pi-i\log(\sqrt{5}-2)$$
$$e^{iy}=-2-\sqrt{5}=(\sqrt{5}+2)e^{\pi i}=(\sqrt{5}+2)e^{\pi i+2n\pi i}, iy=\log(\sqrt{5}+2)+\pi i+2n\pi i, y=\pi+2n\pi-i\log(\sqrt{5}+2)$$
$$\begin{cases} y=2n\pi-i\log(\sqrt{5}-2), \\ y=\pi+2n\pi-i\log(\sqrt{5}+2), \qquad n\in Z \end{cases}$$

(9)

$$\arccos(2i)=y,$$
$$y=\arccos z, z=\cos y=\frac{e^{iy}+e^{-yi}}{2}, X=e^{iy}$$
$$z=\frac{X+X^{-1}}{2}, X^2-2zX+1=0, X=z\pm\sqrt{z^2-1}, z=2i$$
$$e^{iy}=2i\pm\sqrt{-4-1}=2i\pm\sqrt{5}i=(2\pm\sqrt{5})i$$
$$e^{iy}=(2+\sqrt{5})i=(\sqrt{5}+2)i=(\sqrt{5}+2)e^{\frac{\pi}{2}i}=(\sqrt{5}+2)e^{\frac{\pi}{2}i+2n\pi i}, iy=\log(\sqrt{5}+2)+(\frac{\pi}{2}+2n\pi)i$$
$$y=(\frac{\pi}{2}+2n\pi)-i\log(\sqrt{5}+2)$$
$$e^{iy}=(\sqrt{5}-2)(-i)=(\sqrt{5}-2)e^{-\frac{\pi}{2}i}=(\sqrt{5}-2)e^{-\frac{\pi}{2}i+2n\pi i}, iy=\log(\sqrt{5}-2)+(-\frac{\pi}{2}+2n\pi)i$$
$$y=(-\frac{\pi}{2}+2n\pi)-i\log(\sqrt{5}-2)$$
$$\begin{cases} y=(\frac{\pi}{2}+2n\pi)-i\log(\sqrt{5}+2), \\ y=(-\frac{\pi}{2}+2n\pi)-i\log(\sqrt{5}-2), \qquad n\in Z \end{cases}$$

(10)

$\arctan(2i) = y,$

$$\arctan z = y, z = \tan y = \frac{\sin y}{\cos y} = \frac{\dfrac{e^{iy} - e^{-iy}}{2i}}{\dfrac{e^{iy} + e^{-iy}}{2}}, X = e^{iy}$$

$$z = \frac{X - X^{-1}}{i(X + X^{-1})} = \frac{X^2 - 1}{i(X^2 + 1)}, zi(X^2 + 1) = X^2 - 1, (zi - 1)X^2 = -(zi + 1), X^2 = \frac{1 + zi}{1 - zi}, z = 2i$$

$$e^{2yi} = \frac{1 + 2i^2}{1 - 2i^2} = -\frac{1}{3} = \frac{1}{3}e^{i\pi} = \frac{1}{3}e^{i\pi + 2n\pi i}, 2yi = \log(\frac{1}{3}) + (\pi + 2n\pi)i, y = -\frac{1}{2}i(-\log 3) + \frac{\pi + 2n\pi}{2}$$

$$Ans. \quad y = (\frac{\pi}{2} + n\pi) + \frac{\log 3}{2}i, \quad n \in Z$$

（関連問題）

$(8)'$

$$y = \arcsin(1 + i\sqrt{3})$$

$y = \arcsin(z) = -i\log(iz \pm \sqrt{1 - z^2})$ に代入 $y = -i\log\{i(1 + i\sqrt{3}) \pm \sqrt{1 - (1 + i\sqrt{3})^2}\}$

$$y = -i\log(i - \sqrt{3} \pm \sqrt{3 - i2\sqrt{3}}), \sqrt{3 - i2\sqrt{3}} = \sqrt{r_1 e^{i\theta_1}} = \sqrt{r_1 e^{i\theta_1 + 2n_1\pi i}} = \sqrt{r_1}e^{\frac{i(\theta_1 + 2n_1\pi)}{2}}, (n_1 = 0,1)$$

$$r_1 = \sqrt{21}, \theta_1 = -\arctan(\frac{2\sqrt{3}}{3}), \sqrt{3 - i2\sqrt{3}} = (21)^{\frac{1}{4}}e^{\frac{\theta_1}{2}i + n_1\pi i} = \pm(21)^{\frac{1}{4}}e^{\frac{\theta_1}{2}i} = \pm(21)^{\frac{1}{4}}(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}),$$

$$\sqrt{3 - i2\sqrt{3}} = \pm(21)^{\frac{1}{4}}(\sqrt{\frac{1 + \cos\theta_1}{2}} - i\sqrt{\frac{1 - \cos\theta_1}{2}}) = \pm(21)^{\frac{1}{4}}(\sqrt{\frac{1 + \frac{3}{\sqrt{21}}}{2}} - i\sqrt{\frac{1 - \frac{3}{\sqrt{21}}}{2}}),$$

$$y = -i\log[i - \sqrt{3} \pm \{\pm(21)^{\frac{1}{4}}(\sqrt{\frac{1 + \frac{3}{\sqrt{21}}}{2}} - i\sqrt{\frac{1 - \frac{3}{\sqrt{21}}}{2}})\}] = -i\log\{i - \sqrt{3} \pm (\sqrt{\frac{\sqrt{21} + 3}{2}} - i\sqrt{\frac{\sqrt{21} - 3}{2}})\}$$

$$y = -i\log\{(-\sqrt{3} \pm \sqrt{\frac{\sqrt{21} + 3}{2}}) + i(1 \mp \sqrt{\frac{\sqrt{21} - 3}{2}})\}$$

$$y_1 = -i\log\{(-\sqrt{3} + \sqrt{\frac{\sqrt{21} + 3}{2}}) + i(1 - \sqrt{\frac{\sqrt{21} - 3}{2}})\} = -i\log(a_1 + ib_1) = -i\log\{\sqrt{a_1^2 + b_1^2}e^{i\arctan(\frac{b_1}{a_1}) + 2n\pi i}\}$$

$$= -i\{\frac{1}{2}\log(a_1^2 + b_1^2) + i\arctan(\frac{b_1}{a_1}) + 2n\pi i\} = \arctan(\frac{b_1}{a_1}) + 2n\pi - i\frac{1}{2}\log(a_1^2 + b_1^2)$$

$$y_2 = -i\log\{(-\sqrt{3} - \sqrt{\frac{\sqrt{21} + 3}{2}}) + i(1 + \sqrt{\frac{\sqrt{21} - 3}{2}})\} = -i\log(a_2 + ib_2) = -i\log\{\sqrt{a_2^2 + b_2^2}e^{i\arctan(\frac{b_2}{a_2}) + 2n\pi i}\}$$

$$= -i\{\frac{1}{2}\log(a_2^2 + b_2^2) + i\arctan(\frac{b_2}{a_2}) + 2n\pi i\} = \arctan(\frac{b_2}{a_2}) + 2n\pi - i\frac{1}{2}\log(a_2^2 + b_2^2)$$

ここで$a_{1,} b_1, a_2, b_2$を代入する.

Ans.

$$y_1 = \arctan(\sqrt{\frac{2}{3+\sqrt{21}}}) + 2n\pi - i\frac{1}{2}\log\{4+\sqrt{21} - \frac{1}{\sqrt{6}}(3+\sqrt{21})^{\frac{3}{2}}\}$$

$$y_2 = -\arctan(\sqrt{\frac{2}{3+\sqrt{21}}}) + \pi + 2n\pi - i\frac{1}{2}\log\{4+\sqrt{21} + \frac{1}{\sqrt{6}}(3+\sqrt{21})^{\frac{3}{2}}\}, \qquad n \in Z$$

---

(9)′

$y = \arccos(1+i)$

$y = \arccos(z) = -i\log(z \pm \sqrt{z^2-1})$ に代入　$y = -i\log\{(1+i) \pm \sqrt{(1+i)^2-1}\}$

$$y = -i\log(1+i \pm \sqrt{-1+2i}), \sqrt{-1+2i} = \sqrt{r_1 e^{i\theta_1}} = \sqrt{r_1 e^{i\theta_1 + 2n_1\pi i}} = \sqrt{r_1}e^{\frac{i(\theta_1+2n_1\pi)}{2}}, (n_1 = 0,1)$$

$$r_1 = \sqrt{5}, \theta_1 = \pi - \arctan(2), \sqrt{-1+2i} = (5)^{\frac{1}{4}} e^{\frac{\theta_1}{2}i + n_1\pi i} = \pm(5)^{\frac{1}{4}} e^{\frac{\theta_1}{2}i} = \pm(5)^{\frac{1}{4}}(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}),$$

$$\sqrt{-1+2i} = \pm(5)^{\frac{1}{4}}(\sqrt{\frac{1+\cos\theta_1}{2}} + i\sqrt{\frac{1-\cos\theta_1}{2}}) = \pm(5)^{\frac{1}{4}}(\sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} + i\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}),$$

$$y = -i\log[1+i \pm \{\pm(5)^{\frac{1}{4}}(\sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}} + i\sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}})\}] = -i\log\{1+i \pm (\sqrt{\frac{\sqrt{5}-1}{2}} + i\sqrt{\frac{\sqrt{5}+1}{2}})\}$$

$$y = -i\log\{(1 \pm \sqrt{\frac{\sqrt{5}-1}{2}}) + i(1 \pm \sqrt{\frac{\sqrt{5}+1}{2}})\}$$

$$y_1 = -i\log\{(1 + \sqrt{\frac{\sqrt{5}-1}{2}}) + i(1 + \sqrt{\frac{\sqrt{5}+1}{2}})\} = -i\log(a_1 + ib_1) = -i\log\{\sqrt{a_1^2+b_1^2}e^{i\arctan(\frac{b_1}{a_1})+2n\pi i}\}$$

$$= -i\{\frac{1}{2}\log(a_1^2+b_1^2) + i\arctan(\frac{b_1}{a_1}) + 2n\pi i\} = \arctan(\frac{b_1}{a_1}) + 2n\pi - i\frac{1}{2}\log(a_1^2+b_1^2)$$

$$y_2 = -i\log\{(1 - \sqrt{\frac{\sqrt{5}-1}{2}}) + i(1 - \sqrt{\frac{\sqrt{5}+1}{2}})\} = -i\log(a_2 + ib_2) = -i\log\{\sqrt{a_2^2+b_2^2}e^{i\arctan(\frac{b_2}{a_2})+2n\pi i}\}$$

$$= -i\{\frac{1}{2}\log(a_2^2+b_2^2) + i\arctan(\frac{b_2}{a_2}) + 2n\pi i\} = \arctan(\frac{b_2}{a_2}) + 2n\pi - i\frac{1}{2}\log(a_2^2+b_2^2)$$

ここで$a_{1,}b_1, a_2, b_2$を代入する.

Ans.

$$y_1 = \arctan(\sqrt{\frac{\sqrt{5}+1}{2}}) + 2n\pi - i\frac{1}{2}\log\{2+\sqrt{5} + \frac{1}{\sqrt{2}}(\sqrt{5}+1)^{\frac{3}{2}}\}$$

$$y_2 = -\arctan(\sqrt{\frac{\sqrt{5}+1}{2}}) + 2n\pi - i\frac{1}{2}\log\{2+\sqrt{5} - \frac{1}{\sqrt{2}}(\sqrt{5}+1)^{\frac{3}{2}}\}, \qquad n \in Z$$

---

(10)′

$y=\arctan(\sqrt{3}+i)$

$y=\arctan(z)=-\frac{i}{2}\log(\frac{1+zi}{1-zi})$ に代入 $y=-\frac{i}{2}\log\{\frac{1+i(\sqrt{3}+i)}{1-i(\sqrt{3}+i)}\}=-\frac{i}{2}\log(\frac{-3+i2\sqrt{3}}{7})$

$=-\frac{i}{2}\log(a_1+ib_1)=-\frac{i}{2}\log\{\sqrt{a_1^2+b_1^2}e^{i\arctan(\frac{b_1}{a_1})+2n\pi i}\}$

$=-i\{\frac{1}{4}\log(a_1^2+b_1^2)+i\frac{1}{2}\arctan(\frac{b_1}{a_1})+n\pi i\}=\frac{1}{2}\arctan(\frac{b_1}{a_1})+n\pi-i\frac{1}{4}\log(a_1^2+b_1^2)$

ここで$a_{1,}b_1$を代入する.

Ans. $y=-\frac{1}{2}\arctan(\frac{2}{\sqrt{3}})+\frac{1}{2}\pi+n\pi-i\frac{1}{4}\log\frac{3}{7},\quad n\in Z$