数学Ⅲ　レポート　No.5　　解答　　(2016)

1.　(20 点×5=100 点)

(1)

$$J=\oint_C \frac{1}{z^2+z+1}dz, C:|z-i|=1, z^2+z+1=0, z=\frac{-1\pm i\sqrt{3}}{2}, C\text{の境界を含めた}C\text{の内部を}D\text{とする}.$$

$$c_1=\frac{-1+i\sqrt{3}}{2}\in D, J=2\pi i\operatorname*{Res}_{z=c_1} f(z)dz=2\pi i\lim_{z\to c_1}(z-c_1)\times\frac{1}{z^2+z+1}=2\pi i\lim_{z\to c_1}\frac{1}{2z+1}=\frac{2\pi i}{2c_1+1}$$

$$J=\frac{2\pi i}{-1+i\sqrt{3}+1}=\frac{2\sqrt{3}}{3}\pi \qquad \underline{Ans.\quad J=\frac{2\sqrt{3}}{3}\pi}$$

(2)

$$J=\oint_C \frac{1}{z^4+3z^2+2}dz, C:|z-2i|=2, z^4+3z^2+2=(z^2+2)(z^2+1)=0, z=\pm i, \pm\sqrt{2}i,$$

Cの境界を含めたCの内部をDとする.

$$c_1=i\in D, c_2=\sqrt{2}i\in D, N=2, J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j} f(z)dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{(z-c_j)}{z^4+3z^2+2}$$

$$=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{1}{4z^3+6z}=2\pi i\left(\frac{1}{4c_1^3+6c_1}+\frac{1}{4c_2^3+6c_2}\right)=\pi i\left(\frac{1}{2i^3+3i}+\frac{1}{2(\sqrt{2}i)^3+3(\sqrt{2}i)}\right)$$

$$=\pi\left(\frac{1}{-2+3}+\frac{1}{-4\sqrt{2}+3\sqrt{2}}\right)=(1-\frac{\sqrt{2}}{2})\pi \qquad \underline{Ans.\quad J=(1-\frac{\sqrt{2}}{2})\pi}$$

(3)

$$J=\oint_C \frac{z}{z^3+1}dz, C:|z-1-i|=1, z^3+1=0, z^3=-1=e^{\pi i+2n\pi i}, z=e^{\frac{\pi i+2n\pi i}{3}}$$

Cの境界を含めたCの内部をDとする. $z_0=e^{\frac{\pi i}{3}}, z_1=e^{\frac{3\pi i}{3}}, z_2=e^{\frac{5\pi i}{3}}$

$$c_1=e^{\frac{\pi i}{3}}\in D, J=2\pi i\operatorname*{Res}_{z=c_1} f(z)dz=2\pi i\lim_{z\to c_1}\frac{(z-c_1)z}{z^3+1}=2\pi i\lim_{z\to c_1}\frac{2z-c_1}{3z^2}=2\pi i\frac{c_1}{3c_1^2}=\frac{2\pi i}{3c_1}$$

$$=\frac{2\pi i}{3(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})}=\frac{2\pi i}{3(\frac{1}{2}+i\frac{\sqrt{3}}{2})}=\frac{4\pi i}{3(1+i\sqrt{3})}=\frac{\pi i(1-i\sqrt{3})}{3}=\frac{\sqrt{3}}{3}\pi+\frac{\pi}{3}i$$

$$\underline{Ans.\quad J=\frac{\sqrt{3}}{3}\pi+\frac{\pi}{3}i}$$

(4)

$$J=\oint_C \frac{\sin z}{z^2+1}dz, C:|z|=2, z=\pm i, c_0=i, c_1=-i, N=2,$$

$$J=2\pi i\sum_{j=1}^{N}\operatorname*{Res}_{z=c_j} f(z)dz=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{(z-c_j)\sin z}{z^2+1}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{\sin z+(z-c_j)\cos z}{2z},$$

$$J=2\pi i(\frac{\sin c_1}{2c_1}+\frac{\sin c_2}{2c_2})=\pi i\{\frac{\sin i}{i}+\frac{\sin(-i)}{(-i)}\}=2\pi\sin i=2\pi(\frac{e^{i\theta}-e^{-i\theta}}{2i})_{\theta=i}=\frac{2\pi}{2i}(e^{i^2}-e^{-i^2})=\pi(e-\frac{1}{e})i$$

$$\underline{Ans.\quad J=\pi(e-\frac{1}{e})i}$$

(5)

$$J=\oint_C \frac{\cos z}{z^4+1}dz, C:|z-1|=1, z^4=-1=e^{\pi i+2n\pi i}, z=e^{\frac{\pi i+2n\pi i}{4}}, z_0=e^{\frac{\pi i}{4}}, z_1=e^{\frac{3\pi i}{4}}, z_2=e^{\frac{5\pi i}{4}}, z_3=e^{\frac{7\pi i}{4}}$$

$$c_1=e^{\frac{\pi i}{4}}=\frac{\sqrt{2}}{2}(1+i), c_2=e^{\frac{7\pi i}{4}}=\frac{\sqrt{2}}{2}(1-i), N=2$$

$$J=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{(z-c_j)\cos z}{z^4+1}=2\pi i\sum_{j=1}^{N}\lim_{z\to c_j}\frac{\cos z-(z-c_j)\sin z}{4z^3}=2\pi i(\frac{\cos c_1}{4c_1^3}+\frac{\cos c_2}{4c_2^3})$$

$$=2\pi i(\frac{c_1\cos c_1}{4c_1^4}+\frac{c_2\cos c_2}{4c_2^4})$$

$$=-\frac{2\pi i}{4}(c_1\cos c_1+c_2\cos c_2)=-\frac{2\pi i}{4}[\frac{\sqrt{2}}{2}(1+i)\{\frac{e^{i\theta}+e^{-i\theta}}{2}\}_{\theta=\frac{\sqrt{2}}{2}(1+i)}$$

$$+\frac{\sqrt{2}}{2}(1-i)\{\frac{e^{i\theta}+e^{-i\theta}}{2}\}_{\theta=\frac{\sqrt{2}}{2}(1-i)}]$$

$$=-\frac{\sqrt{2}\pi i}{4}\{(1+i)\frac{e^{i\frac{\sqrt{2}}{2}(1+i)}+e^{-i\frac{\sqrt{2}}{2}(1+i)}}{2}+(1-i)\frac{e^{i\frac{\sqrt{2}}{2}(1-i)}+e^{-i\frac{\sqrt{2}}{2}(1-i)}}{2}\}$$

$$=-\frac{\sqrt{2}\pi i}{4}\{(1+i)\frac{e^{-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}+e^{\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}}}{2}+(1-i)\frac{e^{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}+e^{-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}}}{2}\}$$

$$=-\frac{\sqrt{2}\pi i}{8}[(1+i)\{e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2}+i\sin\frac{\sqrt{2}}{2})+e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2}-i\sin\frac{\sqrt{2}}{2})\}$$

$$+(1-i)\{e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2}+i\sin\frac{\sqrt{2}}{2})+e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2}-i\sin\frac{\sqrt{2}}{2})\}]$$

$$= -\frac{\sqrt{2}\pi i}{8}\{e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2}) + ie^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2})$$
$$+ e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}) + ie^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2})$$
$$+ e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}) + ie^{\frac{\sqrt{2}}{2}}(-\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2})$$
$$+ e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2}) + ie^{-\frac{\sqrt{2}}{2}}(-\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2})\}$$

$$= -\frac{\sqrt{2}\pi i}{8}\{e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2}) + e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2})$$
$$+ e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}) + e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2})$$
$$+ ie^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2}) + ie^{\frac{\sqrt{2}}{2}}(-\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}) +$$
$$ie^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}) + ie^{-\frac{\sqrt{2}}{2}}(-\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2})\}$$
$$= -\frac{\sqrt{2}\pi i}{4}\{e^{-\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} - \sin\frac{\sqrt{2}}{2}) + e^{\frac{\sqrt{2}}{2}}(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2})\}$$
$$= -\frac{\sqrt{2}\pi}{2}(\cos\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}\sinh\frac{\sqrt{2}}{2})i$$

Ans. $J = -\frac{\sqrt{2}\pi}{2}(\cos\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2}\sinh\frac{\sqrt{2}}{2})i$

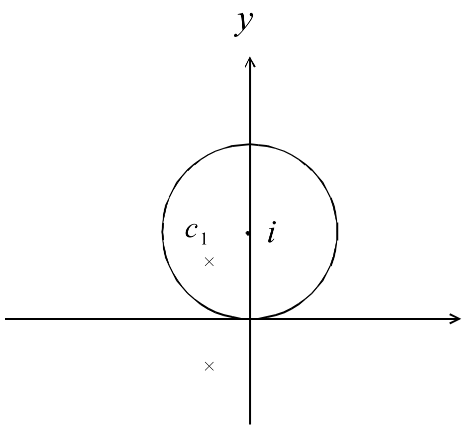

Fig.1 (1)

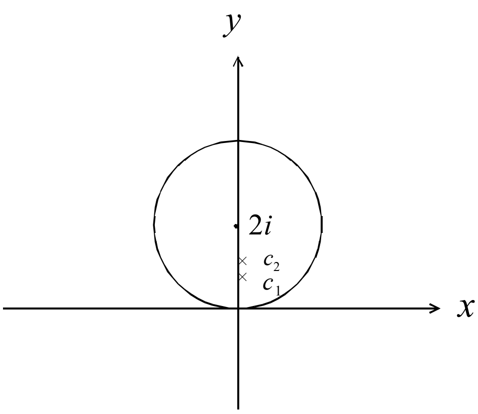

Fig.2 (2)

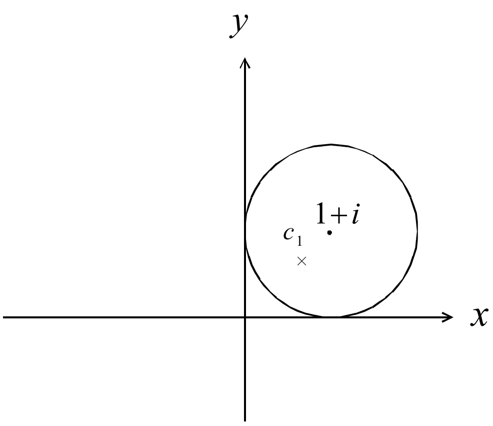

Fig .3 (3)

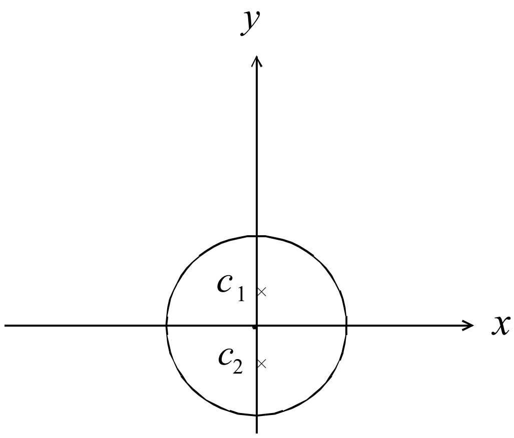

Fig.4 (4)

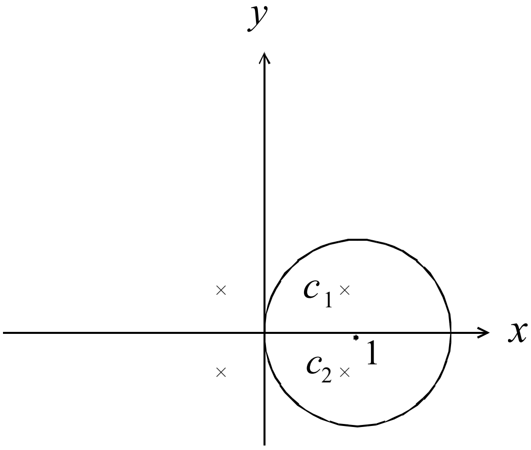

Fig.5 (5)